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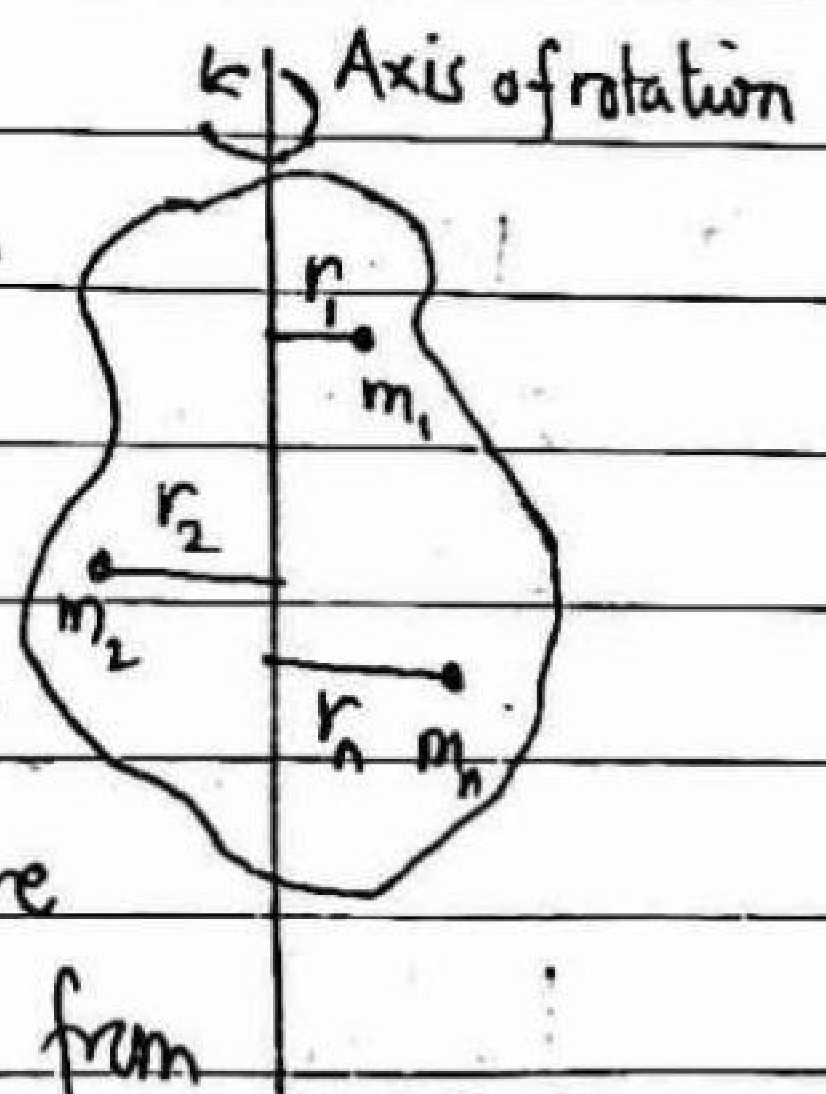
Dynamics Of Rigid Bodies

A rigid body is that in which relative separation between constituent particles remains same under the application of force or forces. So, rigid body has fixed shape irrespective of its state of rest or motion as well as under the action of external force or forces.

Moment of inertia: The property of a material body by virtue of which it opposes any change in rotational motion is called moment of inertia. So moment of inertia is rotational analogy of inertia in linear dynamics.



Mathematically, moment of inertia dended by I is defined as the sum of products of masses of constituent particles and square of perpendicular distances from the axis of rotation.

If a body comprises constituent particles of masses $m_1, m_2, \ldots m_n$ situated at perpendicular distances $r_1, r_2, \ldots r_n$ from given axis of rotation then its moment of inertia, I can be expressed as;

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$$I = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2$$

or, $$\boxed{I = \sum_{i=1}^{n} m_i r_i^2}$$

* For a point particle of mass 'm' lying at perpendicular distance r from axis of rotation then its moment of inertia becomes,

$$\boxed{I = mr^2}$$

SI unit of moment of inertia is $kg\,m^2$ and dimensional formula is $[ML^2]$

* Moment of inertia of a given rigid body about a fixed axis of rotation is constant but it may change for a same body when axis of rotation changes.

Radius of Gyration: Radius of gyration denoted by K is defined as the perpendicular distance from given axis of rotation to a point where entire mass of a given body can be supposed to be located so that moment of inertia remains same as that of in original distribution of particles of the given body.

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If a body consists of particles $m_1, m_2, \ldots m_n$ at perpendicular distances $r_1, r_2, \ldots r_n$ from axis of rotation then its moment of inertia is given by;

$$I = (m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2) \quad \text{--- (1)}$$

Further, if M be the entire mass of the body and K be the radius of gyration then from definition

$$I = MK^2 \quad \text{--- (2)} \Rightarrow \boxed{K = \sqrt{\frac{I}{M}}}$$

Equating (1) and (2) we get;

$$MK^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2$$

or, $$K^2 = \frac{m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2}{M}$$

or, $$K^2 = \frac{m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2}{(m_1 + m_2 + \cdots + m_n)}$$

Considering, $m_1 = m_2 = \cdots = m_n = m$ we have;

$$K^2 = \frac{m(r_1^2 + r_2^2 + \cdots + r_n^2)}{(m + m + \cdots + m)}$$

or, $$K^2 = \frac{\cancel{m}(r_1^2 + r_2^2 + \cdots + r_n^2)}{n\cancel{m}}$$

or, $$K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}} \quad \text{--- (3)}$$

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So, radius of gyration can also be defined as the root mean square distance of the identical constituent particles of the given body from given axis of rotation.

Theorem of parallel axes:

It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of total mass of the body and the square of distance between two parallel axes. In short;

$$\boxed{I = I_{cm} + Mr^2}$$

where I is total moment of inertia about any arbitrary axis, I_{cm} is moment of inertia of same body about an axis passing through centre of mass of the given body being parallel to given arbitrary axis, M is total mass of the body and r is the perpendicular distance between two parallel axes.

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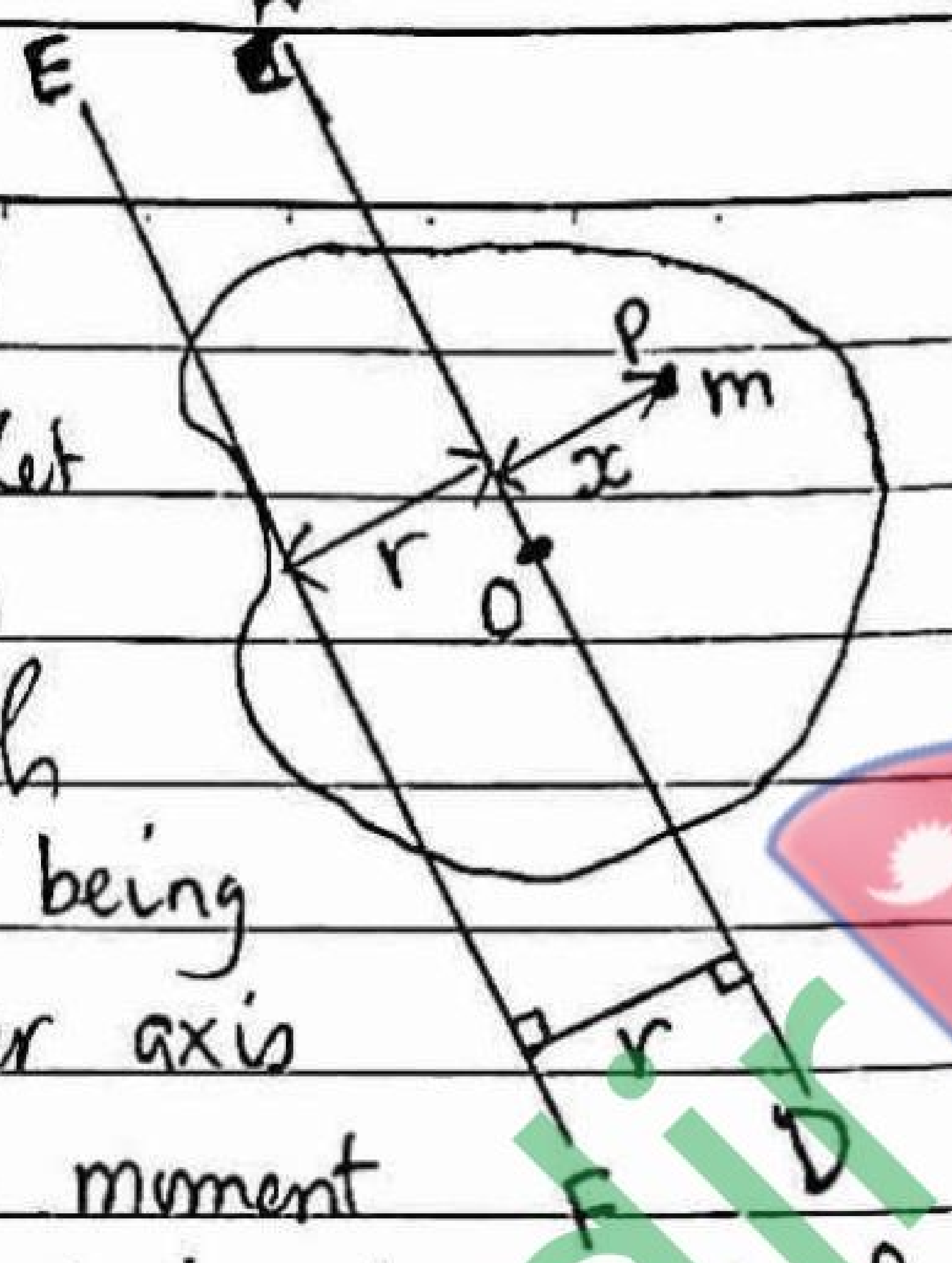
Proof



Consider a rigid body of mass M. Let CD be an axis that passes through centre of mass 'O' being parallel to another axis EF about which moment of inertia of the body is required.

Let's take any constituent particle at P having mass 'm' and lying at perpendicular distance x from axis CD (axis through centre of mass) then moment of inertia of this particle from another axis EF is $m(r+x)^2$.

So, total moment of inertia of entire rigid body about axis EF is given by

$$I = \sum m(r+x)^2$$

or, $I = \sum m(r^2 + x^2 + 2rx)$

or, $I = \sum mr^2 + \sum mx^2 + \sum 2rmx$

or, $I = r^2 \sum m + I_{cm} + 2r\sum mx$

or, $I = I_{cm} + Mr^2 + 2r \times 0$

or, $\boxed{I = I_{cm} + Mr^2}$ This proves theorem of parallel axes.

Here, $\sum mx^2 = I_{cm}$ (moment of inertia about an axis through CM)

$\sum mx = 0$ (moment due to mass about axis through CM is always zero)

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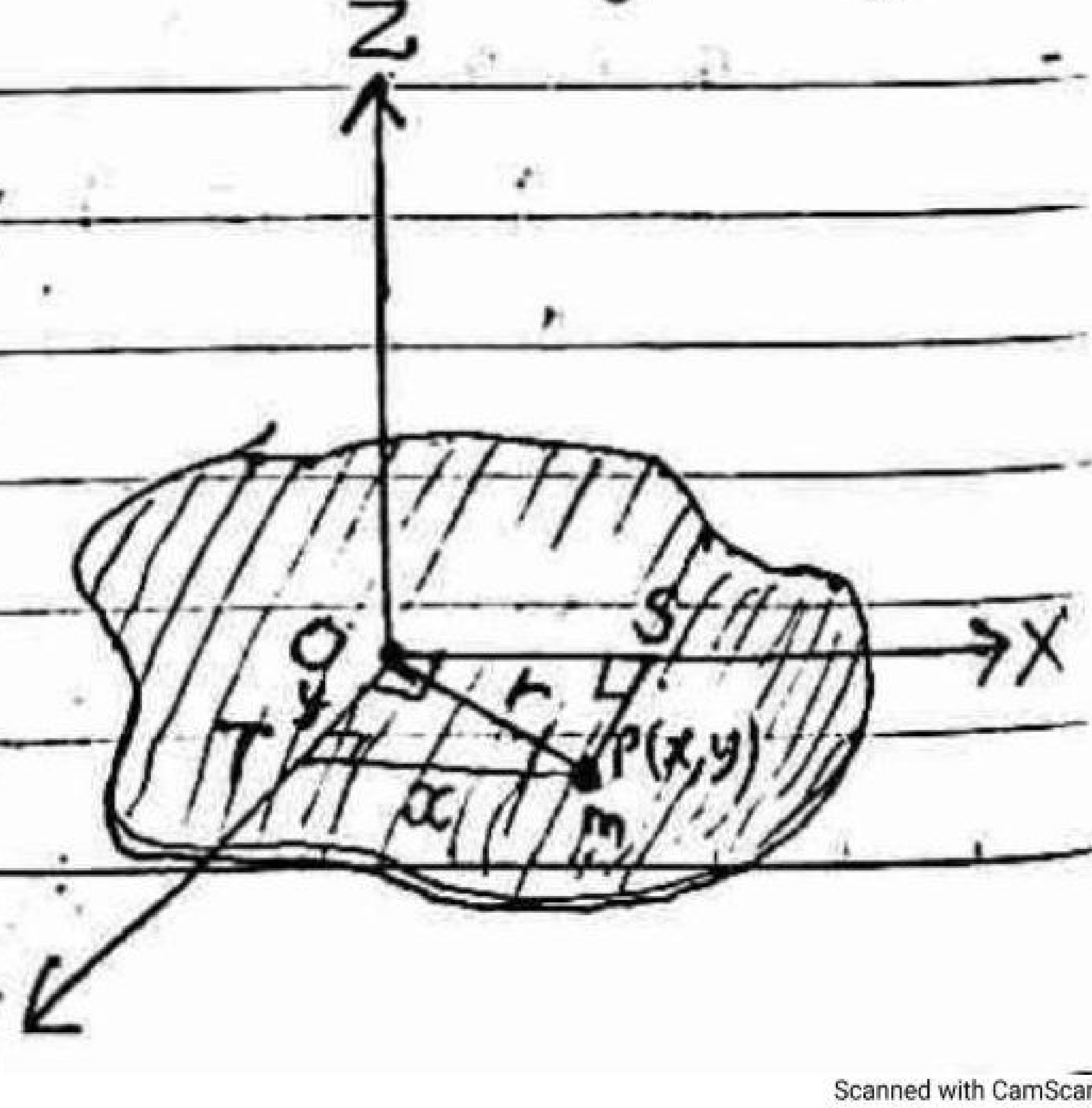
Theorem of perpendicular axes

The theorem states that the sum of moments of inertia of a laminar body about any two mutually perpendicular axes on its plane is equal to the moment of inertia of the body about another axis perpendicular to the plane and passing through point of intersection of the two axes.

If I_x and I_y are the moments of inertia of a laminar body about two mutually perpendicular axes 'X' and 'Y' on the plane of lamina intersecting at point 'O' then moment of inertia of the body about another axis Z perpendicular to the plane of lamina and passing through point 'O' according to perpendicular axes theorem is given by

$$I_Z = I_X + I_Y$$



Proof. Let's take a lamina on X-Y plane such that OX and OY lie on the plane

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and OZ passes being perpendicular to the plane of the lamina through point of intersection 'O'.

If particle at P has mass 'm' situated at perpendicular distance r from axis OZ then moment of inertia of the particle about axis OZ becomes mr^2. So moment of inertia of entire lamina about axis OZ can be expressed as,

$$I_z = \sum mr^2 \quad \text{--- (1)}$$

Similarly, moment of inertia of the entire lamina about axis OX is given by

$$I_x = \sum m(SP)^2 = \sum my^2$$

And, moment of inertia of entire lamina about axis OY is given by

$$I_y = \sum m(TP)^2 = \sum mx^2$$

$$\therefore I_x + I_y = \sum mx^2 + \sum my^2$$

or, $I_x + I_y = \sum m(x^2+y^2)$

or, $I_x + I_y = \sum mr^2$ $\quad [\because x^2+y^2 = r^2]$

or, $I_x + I_y = I_z$ --- (2) [Using (1)]

$\therefore I_x + I_y = I_z$

This ~~prop~~ proves perpendicular axes theorem.

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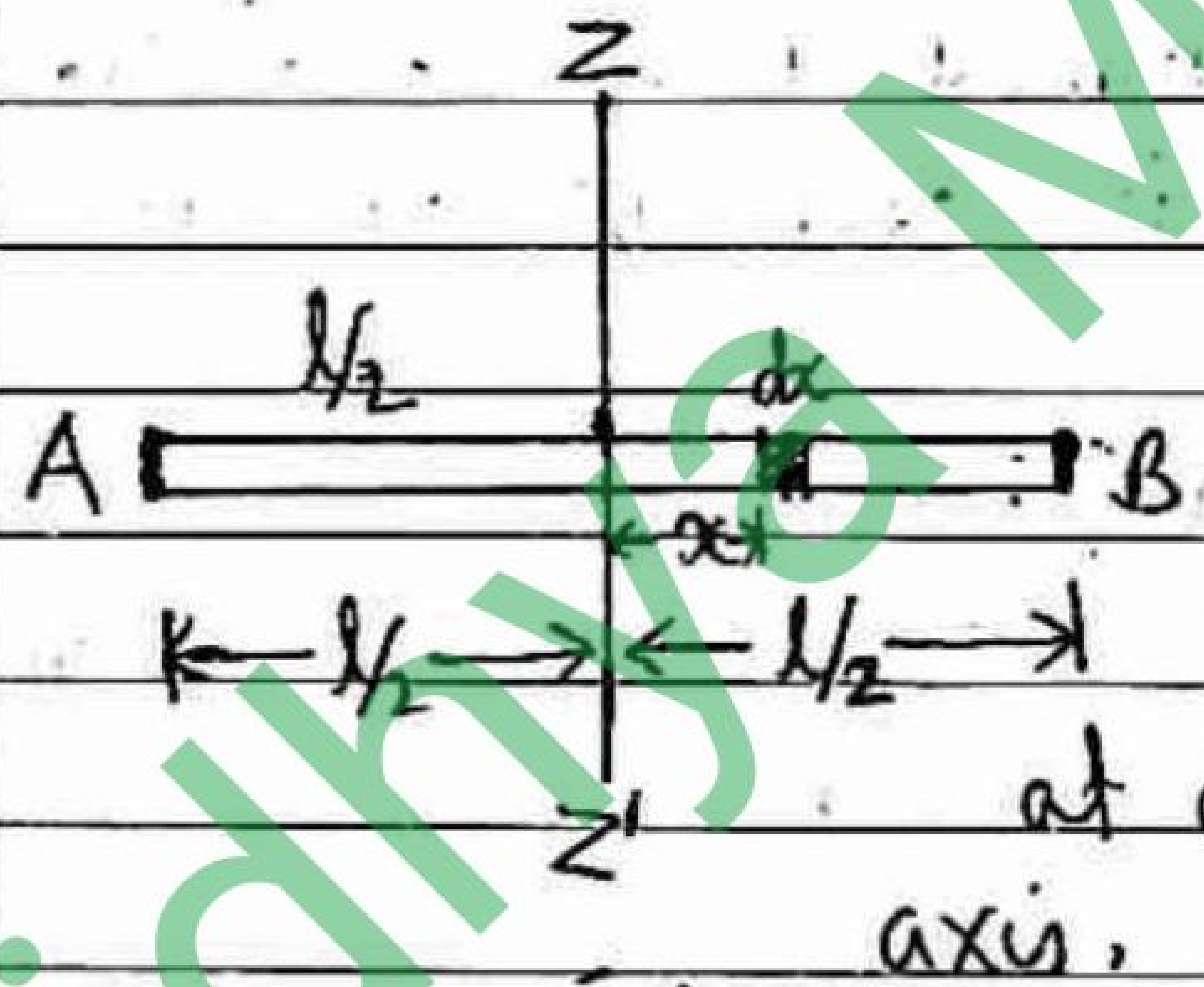
Moment of inertia of uniform shaped bodies:

1. Moment of inertia of a thin uniform rod/bar:

(a) About an axis being perpendicular bisector of its length:

Consider a uniform and thin rod or bar of length l and mass M with axis of rotation ZZ' that passes through mid point being perpendicular to the length of the rod or bar as shown in figure,



Let's take an elementary segment of the rod having length $dx \to 0$ and at distance x from the axis. So

mass of the elementary segment $(dm) = \frac{M}{l} dx$

moment of inertia of elementary segment

$$dI = dm\, x^2$$

or, $$dI = \left(\frac{M}{l} dx\right) x^2$$

or, $$dI = \frac{M}{l} x^2 dx \quad \text{——} \quad ①$$

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The moment of inertia of the entire rod can be found by integrating equation (1) within limits $x=0$ to $x=l/2$ and multiplying by 2, i.e.,

$$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx$$

or, $$I = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

or, $$I = \frac{2M}{3l}\left(l/2\right)^3$$

or, $$I = \frac{2M}{3l} \times \frac{l^3}{8}$$

or, $$\boxed{I = \frac{Ml^2}{12}} \quad \text{——— (2)}$$

This gives moment of inertia of a thin and uniform rod/bar about an axis through mid point of the bar being perpendicular to its length.

If K be the radius of gyration then

$$MK^2 = I = \frac{Ml^2}{12}$$

or, $$K = \sqrt{\frac{l^2}{12}} = \frac{l}{2\sqrt{3}}$$

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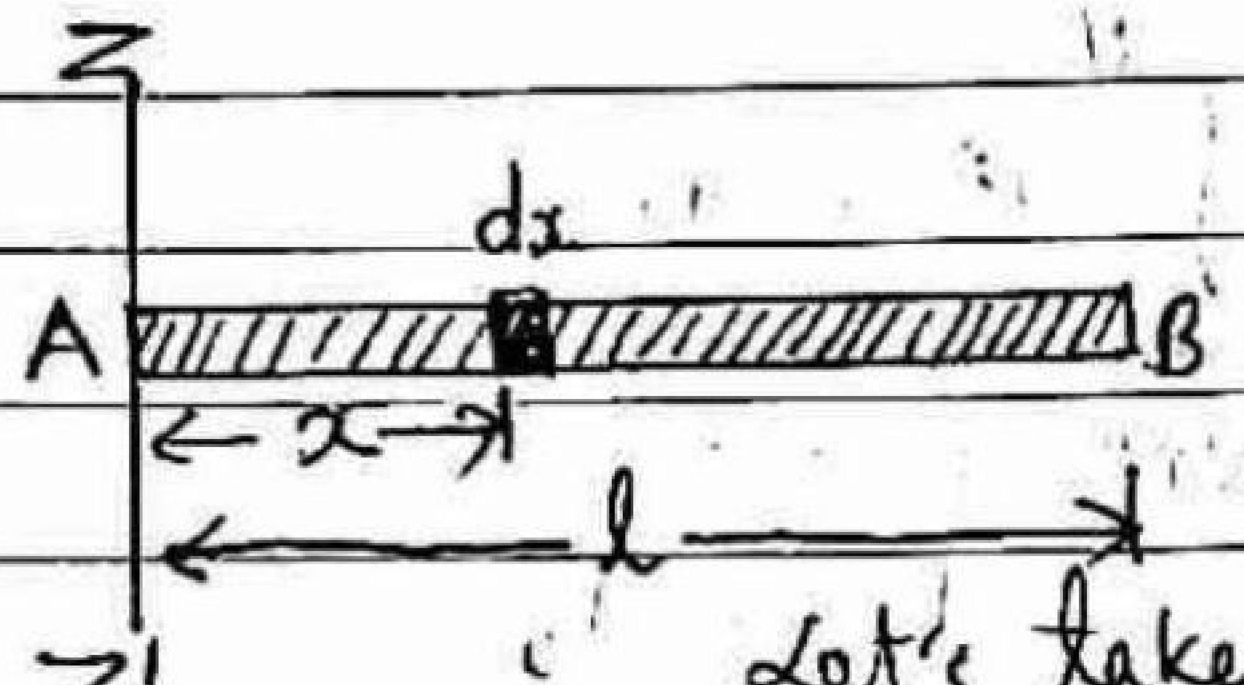
(b) About an axis through one end of the rod being perpendicular to its length:

The situation is shown in figure;



Let's take an elementary segment of length $dx \to 0$ at perpendicular distance x from axis ZZ' so that,

mass of elementary segment $(dm) = \frac{M}{l} dx$

moment of inertia of elementary segment,

$$dI = (dm)\, x^2$$

or, $dI = \frac{M}{l} dx\, x^2$

or, $dI = \frac{M}{l} x^2 dx$ —— ①

Now, moment of inertia of entire rod can be found by integrating equation ① within limits $x = 0$ to $x = l$, i.e.,

$$I = \int_0^l \frac{M}{l} x^2 dx$$

or, $I = \frac{M}{l} \int_0^l x^2 = \frac{M}{l} \left[\frac{x^3}{3}\right]_0^l = \frac{Ml^2}{3}$

or, $\boxed{I = \frac{Ml^2}{3}}$ —— ②

If K be the radius of gyration of the thin rod about given axis then

$$MK^2 = I = \frac{Ml^2}{3}$$

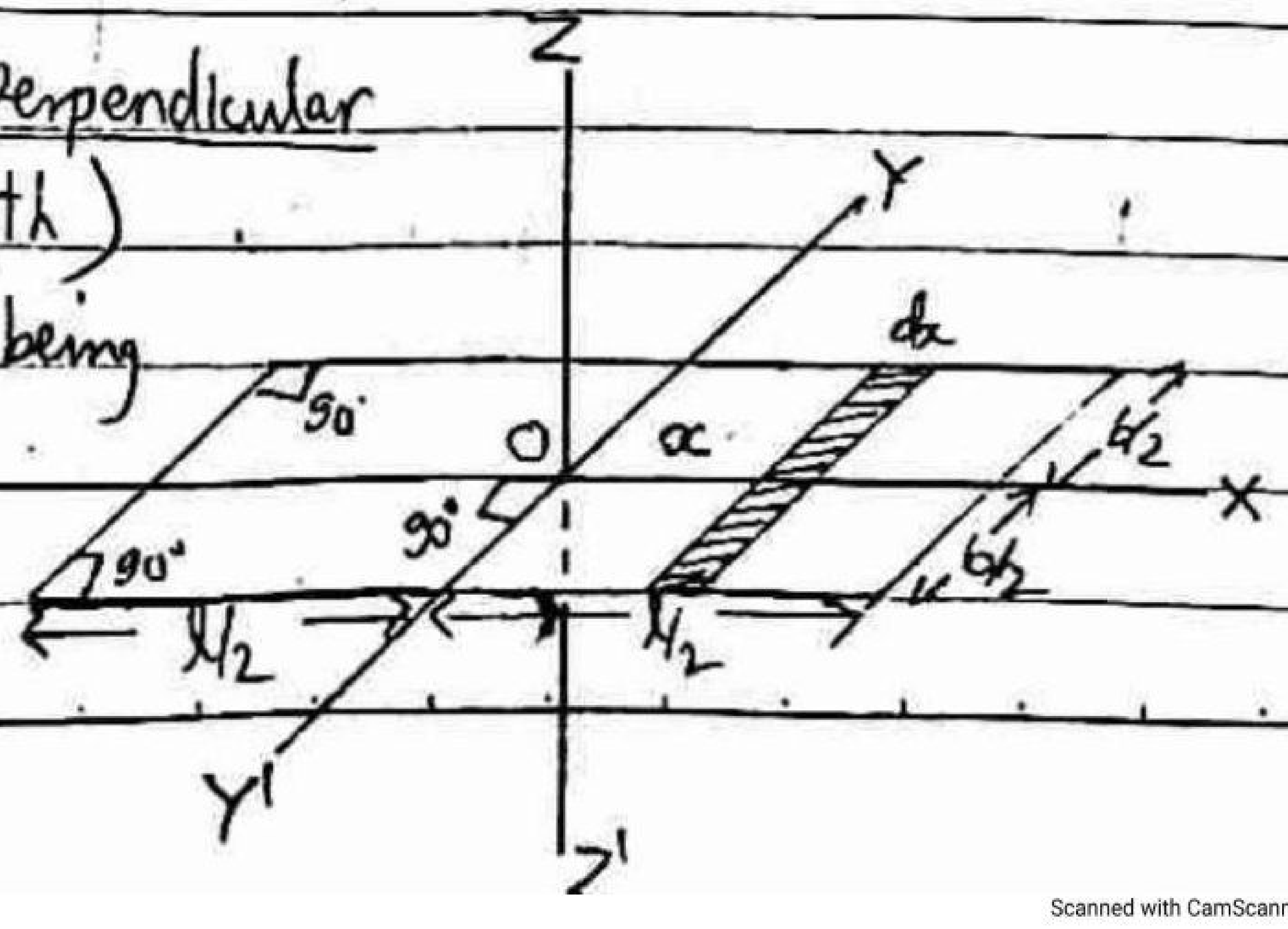
$$\therefore \boxed{K = \frac{l}{\sqrt{3}}}$$

2. <u>Moment of inertia of a rectangular lamina</u>

Consider a rectangular lamina of mass 'M' having length 'l' and breadth 'b' lying on X-Y plane with origin 'O' at the centre of lamina such that X-axis passes parallel to length, Y axis passes parallel to breadth and Z axis passes being perpendicular to the plane of the lamina through point of intersection of X and Y axes (origin) as shown in figure.

(a) <u>About Y-axis (perpendicular bisector of length)</u>

Y axis passes being perpendicular bisector of length and parallel to breadth.



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Let take and elementary strip of width dx at distance x from Y-axis so that area of elementary strip becomes;

$$dA = b \times dx$$

mass of elementary strip $(dm) = \frac{M}{(l \times b)} \times (b \times dx)$

$$\text{or, } (dm) = \frac{M}{l} dx$$

moment of elementary strip $(dI) = dm\, x^2$

$$\text{or, } dI = \frac{M}{l} x^2 dx \text{ --- } (1)$$

Now, total moment of inertia of entire lamina about Y-axis can be found by integrating equation (1) within limits $x = 0$ to $x = l/2$ and multiplying by 2 i.e.

$$I_Y = 2 \int_0^{l/2} \frac{M}{l} x^2 dx$$

$$\text{or, } I_Y = 2 \frac{M}{l} \left[\frac{x^3}{3} \right]_0^{l/2}$$

$$\text{or, } \boxed{I_Y = \frac{Ml^2}{12}}$$

(b) Moment of inertia of the lamina about X-axis (perpendicular bisector of breadth):

X-axis is perpendicular bisector of breadth and parallel to length. So, by similar process,

$$\boxed{I_x = \frac{Mb^2}{12}}$$

(c) About Z-axis:

Since Z-axis is perpendicular to plane of lamina and passes through point of intersection of X and Y axes that lie on the plane of lamina, the moment of inertia of the lamina about Z-axis, according to perpendicular axes theorem can be expressed as,

$$I_z = I_x + I_y$$

or, $$I_z = \frac{Mb^2}{12} + \frac{Ml^2}{12}$$

or, $$\boxed{I_z = M\left(\frac{l^2+b^2}{12}\right)}$$

3. Moment of inertia of thick uniform bar

Consider a uniform bar of rectangular cross-section having mass 'M', length 'l', breadth 'b' and thickness t as shown in figure;

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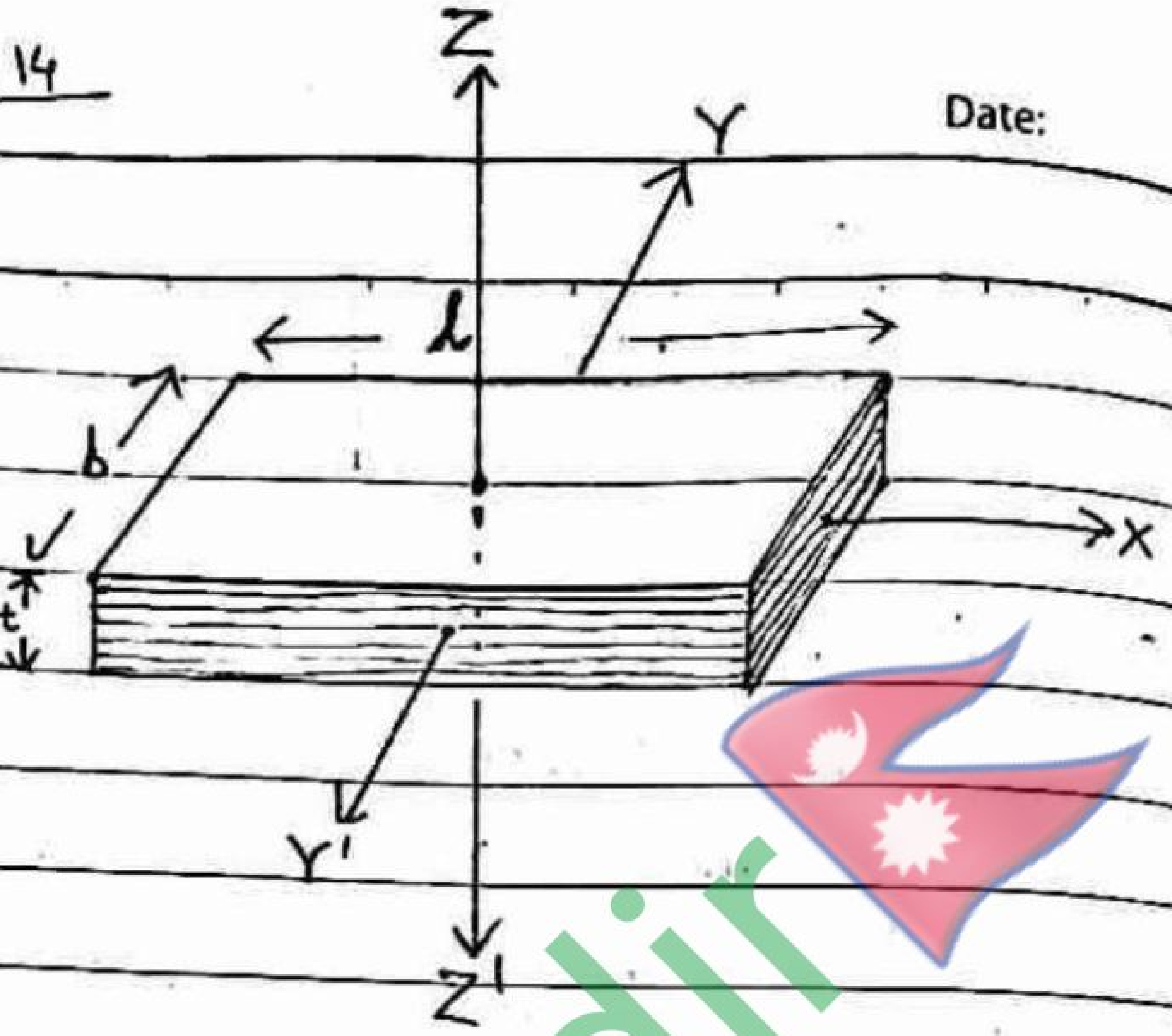

(a) About Z axis (perpendicular to the face $l \times b$ and passing through centre of mass):

The bar can be supposed to be made of a pile of large number of laminas. Let 'm' be the mass of each such lamina. For given axis the lamina has length l and breadth b and Z-axis passes being perpendicular to the plane of the lamina. So, moment of inertia of each lamina about Z-axis becomes $m\left(\frac{l^2+b^2}{12}\right)$

Therefore total moment of inertia of entire bar about Z-axis becomes

$$I_Z = \sum m\left(\frac{l^2+b^2}{12}\right)$$

or, $$\boxed{I_Z = M\left(\frac{l^2+b^2}{12}\right)}$$

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(b) About Y-axis (perpendicular to the face $l \times t$ and passing through centre of mass)

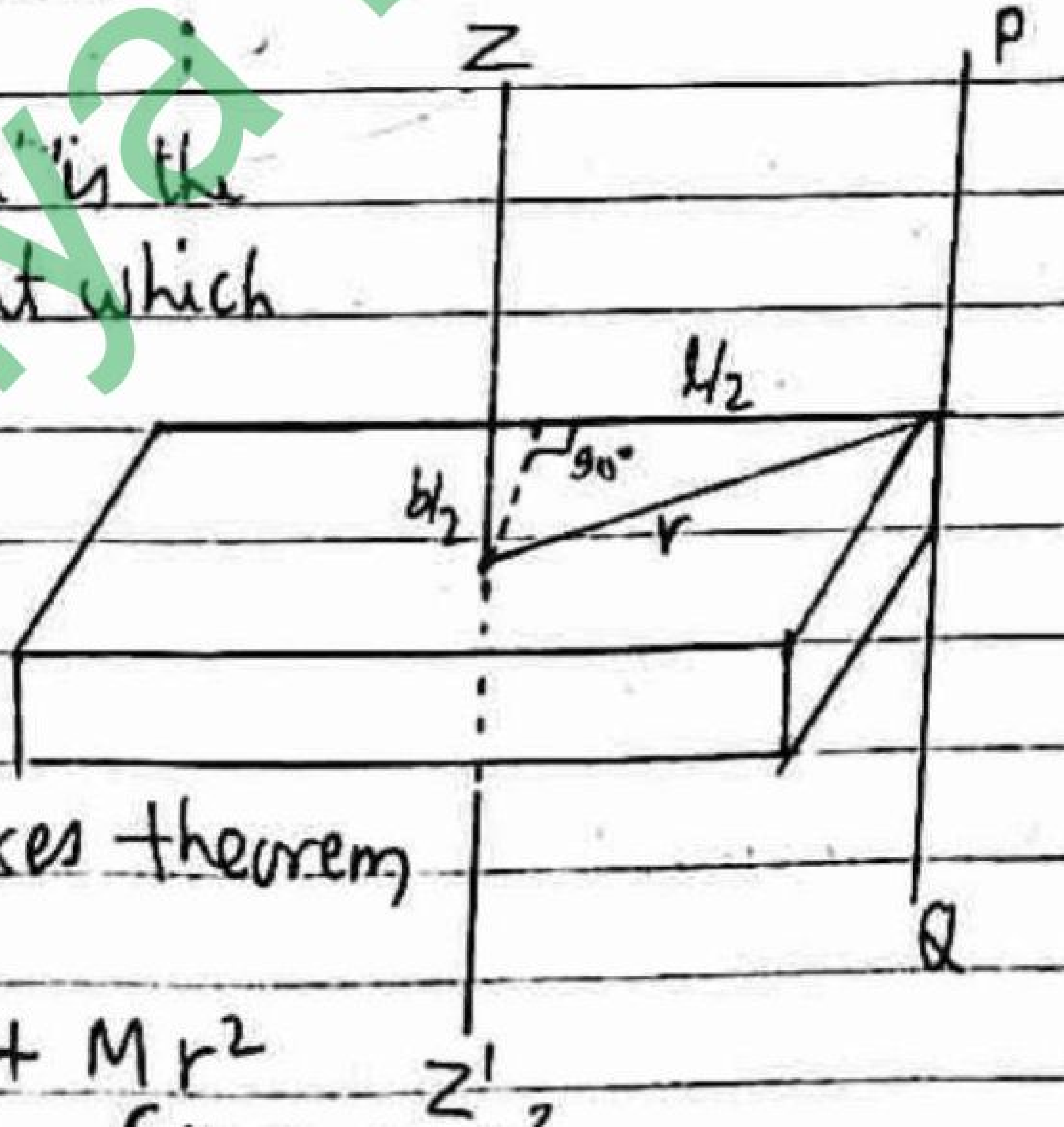
By similar analogy $$\boxed{I_Y = M\left(\frac{l^2+t^2}{12}\right)}$$

(c) About X-axis (perpendicular to face $b \times t$)

$$\boxed{I_X = M\left(\frac{b^2+t^2}{12}\right)}$$

(d) About an axis perpendicular to length of the bar and passing through one of its corners being parallel to Z-axis.

Here PQ is the axis about which moment of inertia is required.



So, from parallel axes theorem

$$I = I_{cm} + Mr^2$$

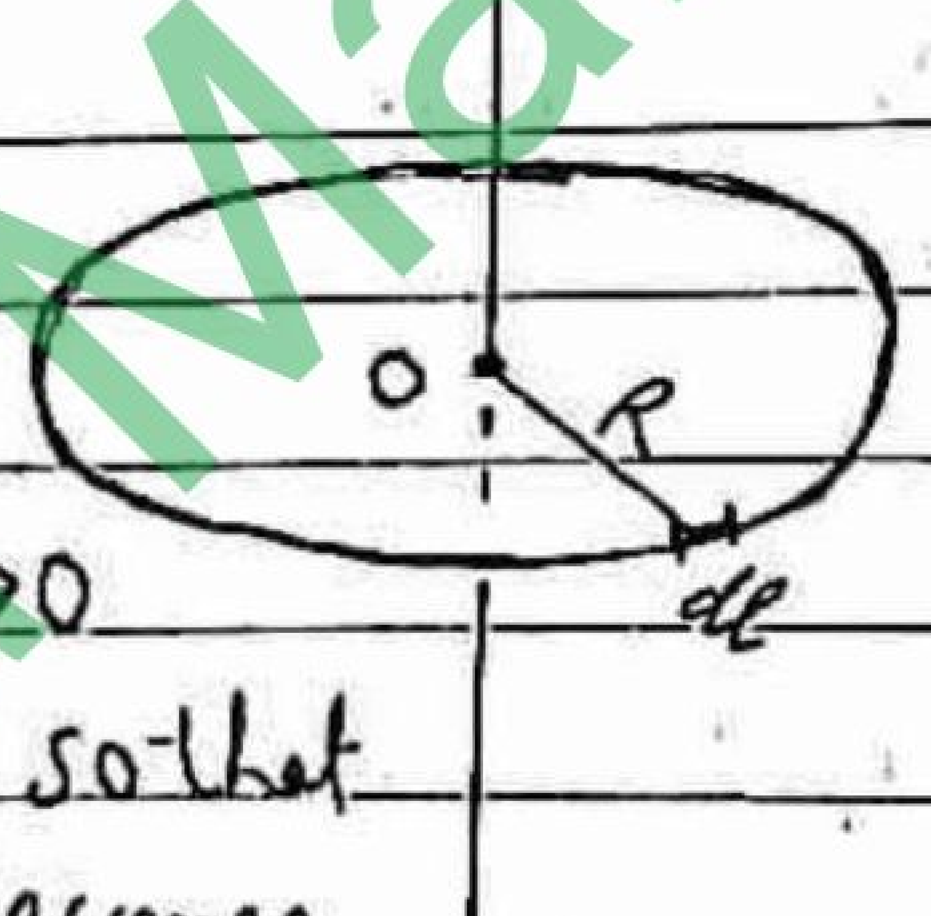
$$\therefore I = I_z + M\left\{(l/2)^2 + (b/2)^2\right\}$$

$$\therefore I = M\left(\frac{l^2+b^2}{12}\right) + M\left(\frac{l^2+b^2}{4}\right) = M\left(\frac{l^2+b^2}{3}\right)$$

4. Moment of inertia of a thin circular ring:

(a) About an axis through its centre being perpendicular to the plane of the ring:

Consider a thin ring of radius R centred at 'O' having mass M and ZZ' be the axis that passes through the centre of the ring being perpendicular to the plane of the ring as shown in figure:



Let's take an elementary segment dl→0 of the ring so that its mass becomes,

$$dm = \frac{M}{2\pi R} dl$$

So, moment of inertia becomes,

$$dI = dm\, R^2$$

or, $dI = \left(\frac{M}{2\pi R} dl\right) R^2 = \frac{MR}{2\pi} dl$ —(1)

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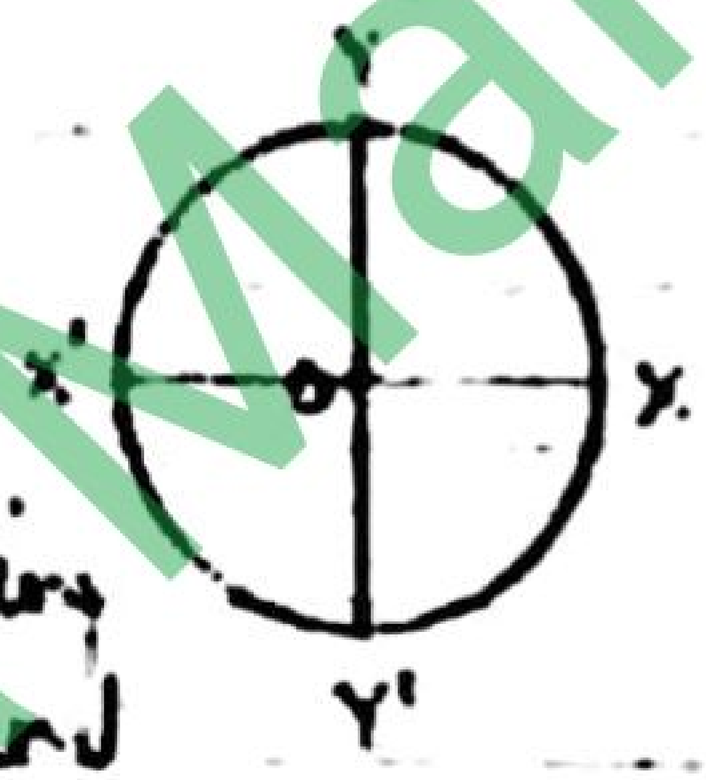
Now total moment of inertia of the entire ring can be found by integrating equation (1) within limits 0 to $2\pi R$ i.e.

$$I_z = \int_0^{2\pi R} \frac{MR}{2\pi} dl$$

or, $I_z = \frac{MR}{2\pi} [l]_0^{2\pi R}$

or, $\boxed{I_z = MR^2}$

(b). About any diametrical axis



If I_x and I_y are moment of inertia of the ring about X axis and Y axis respectively then from symmetry

$$I_x = I_y = I_D \text{ (About diameter)}$$

But, from perpendicular axes theorem

$$I_z = I_x + I_y$$

or, $MR^2 = I_x + I_x \quad [\because I_x = I_y$

or, $MR^2 = 2I_x = 2I_D$

or, $\boxed{I_D = \frac{MR^2}{2}}$

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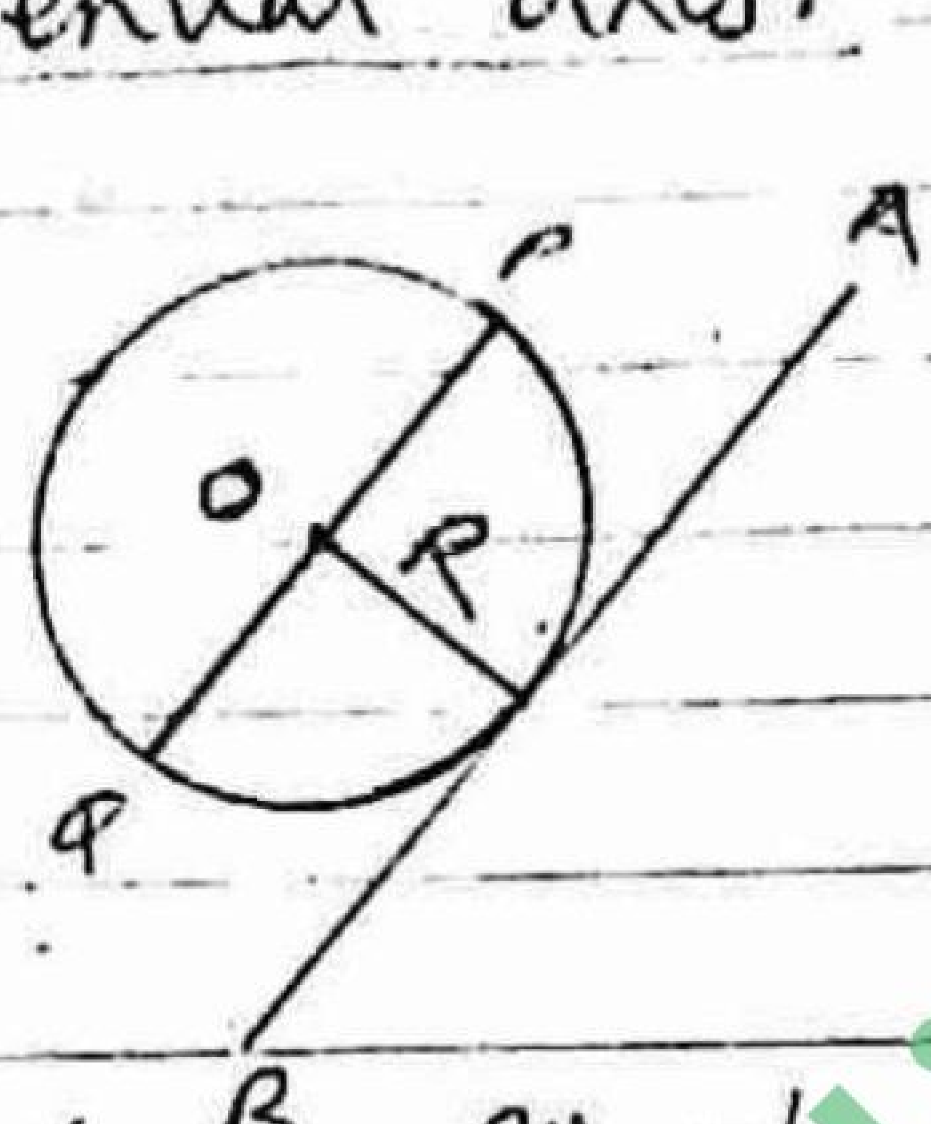
(c) About tangential axis:



Let PQ be any diameter of the ring and AB be a tangential axis parallel to diameter PQ about which moment of inertia is required. If I_T be the moment of inertia of the ring about tangential axis and I_D be its moment of inertia about diameter then from ∵ parallel axes theorem,

$$I_T = I_D + MR^2$$

$$\text{or, } I_T = \frac{MR^2}{2} + MR^2$$

$$\text{or, } \boxed{I_T = \frac{3}{2} MR^2}$$

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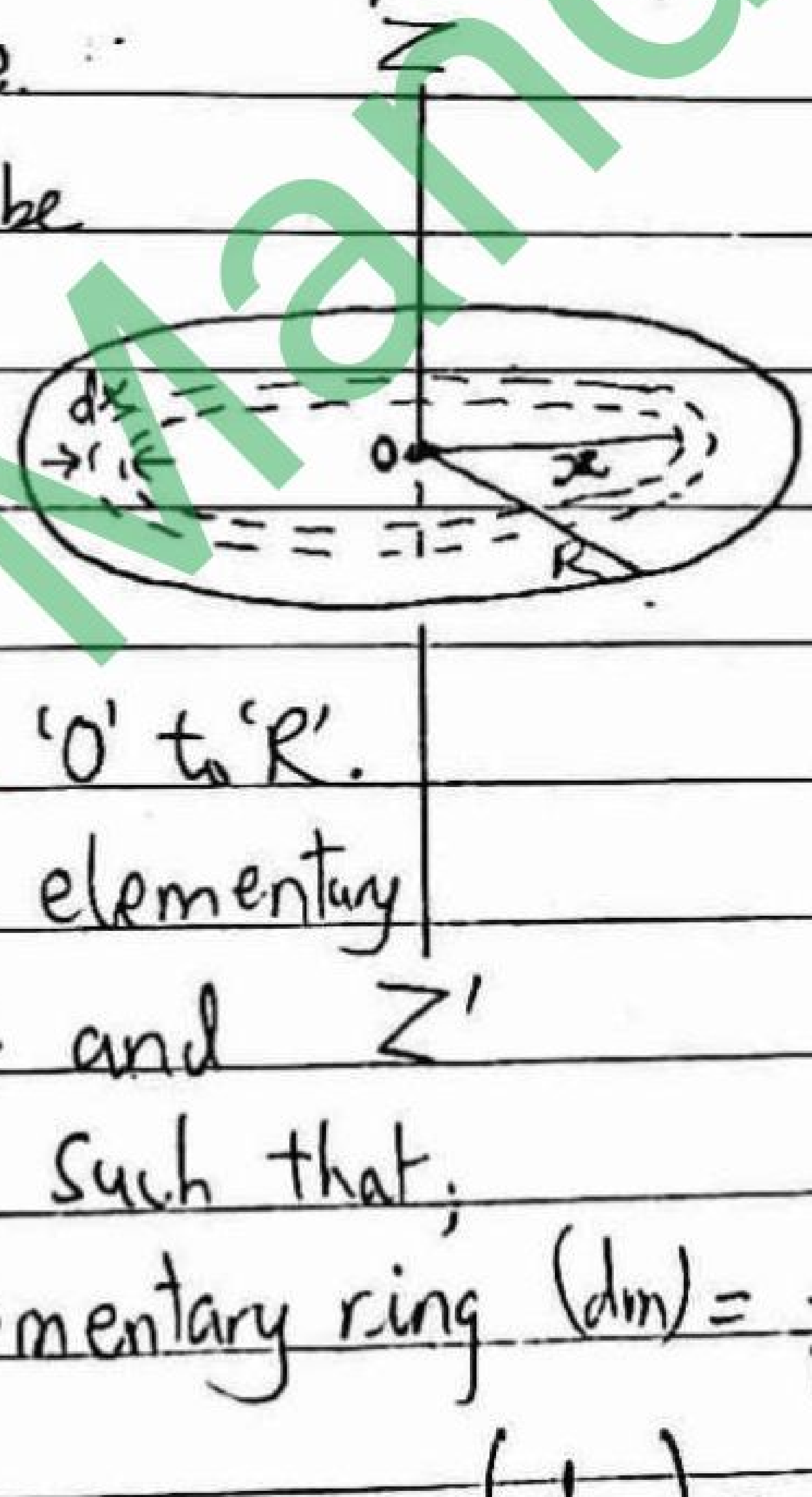
5. Moment of inertia of a circular disc

(a) About an axis through centre being perpendicular to the plane of the disc.

Consider a circular disc of mass M, radius R centred at O and ZZ' be the axis of rotation that passes through centre 'O' being perpendicular to the plane of the disc as shown in figure.



The disc can be supposed to be made up of concentric rings having radii from 'O' to 'R'.

Let's take an elementary ring of radius x and width $dx \to 0$ such that;

mass of elementary ring $(dm) = \frac{M}{\pi R^2}(2\pi x\, dx)$

or, $(dm) = \frac{2Mx}{R^2}dx$

moment of inertia of elementary ring about axis ZZ' becomes $(dI) = dm\, x^2$

or, $dI = \frac{2Mx^3}{R^2}dx$ ——— (1)

The moment of inertia of entire ring about the given axis can be found by integrating equation ① within limits $x=0$ to $x=R$ i.e.

$$I_Z = \int_0^R \frac{2Mx^3}{R^2}\,dx$$

or, $$I_Z = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

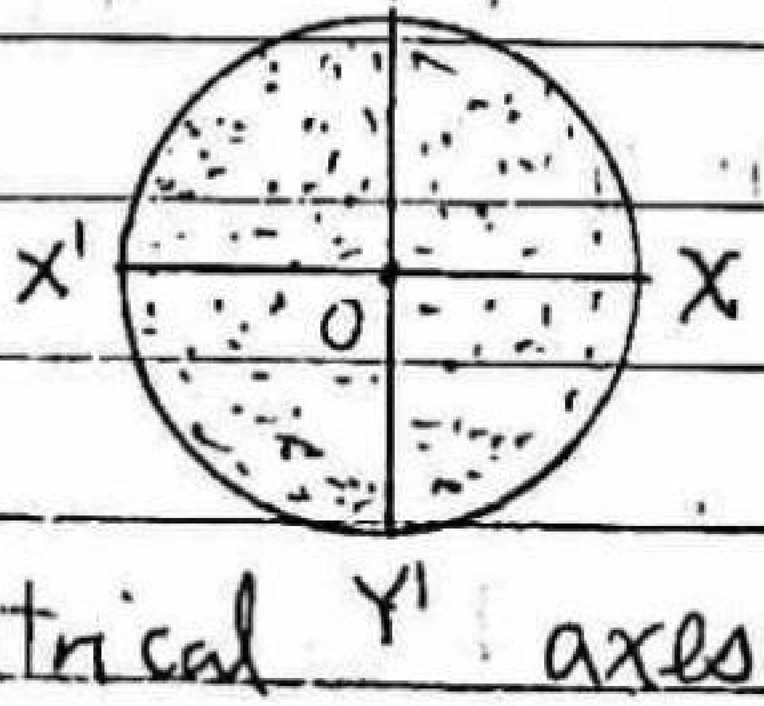
or, $$\boxed{I_Z = \frac{1}{2}MR^2} \quad \text{—— ②}$$

This gives moment of inertia of the disc about axis ZZ'.

If K be the radius of gyration about given axis then

$$K = \sqrt{I/M} = \sqrt{\frac{\frac{1}{2}MR^2}{M}} = \frac{R}{\sqrt{2}}$$

(b) About any diametrical axis



If I_X and I_Y are the moment of inertia of the disc about diametrical axes XX' and YY' respectively then from symmetry; $I_X = I_Y$

Also, from perpendicular axes theorem

$$I_X + I_Y = I_Z$$

or, $$I_X + I_X = \frac{1}{2}MR^2 \Rightarrow \boxed{I_X = I_Y = \frac{MR^2}{4}}$$

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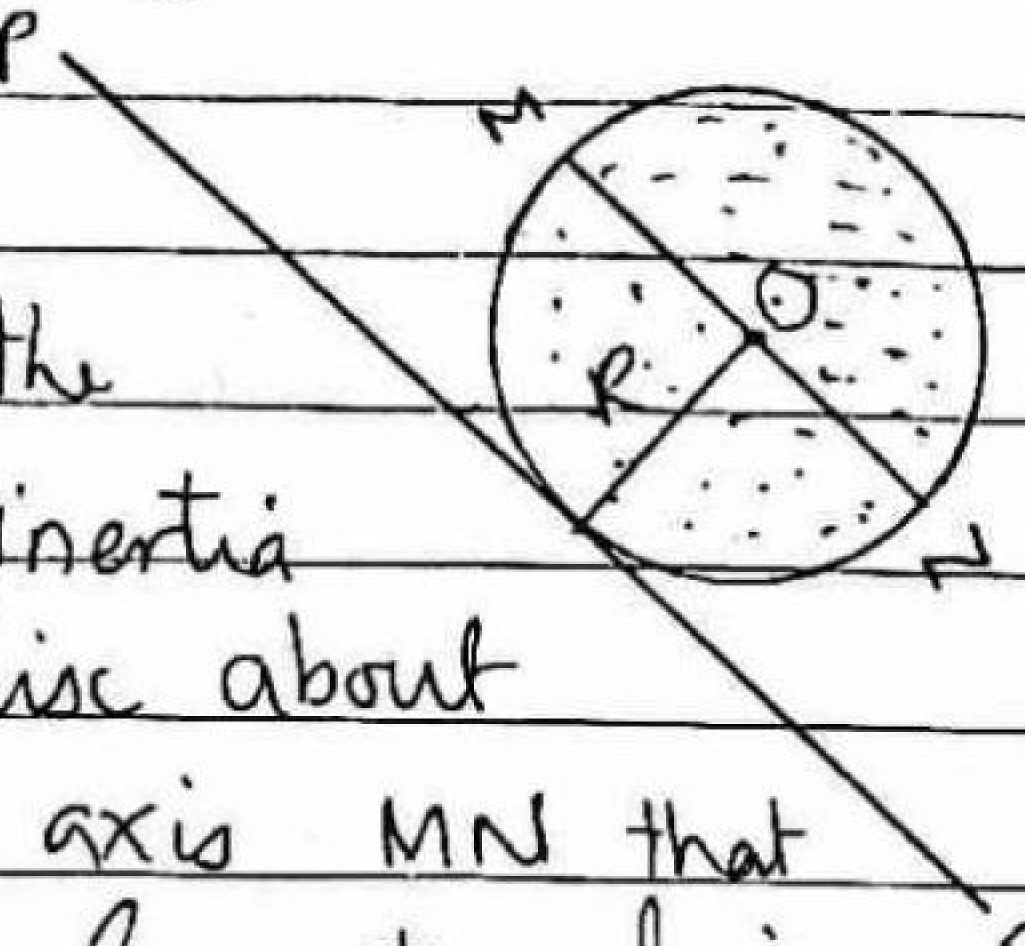
(c) About tangential axis:



Let I_D be the moment of inertia of given disc about diametrical axis MN that passes through centre lying on the plane of page (disc), So, moment of inertia about Tangent PQ, according to parallel axes theorem can be expressed as;

$$I_T = I_D + MR^2$$

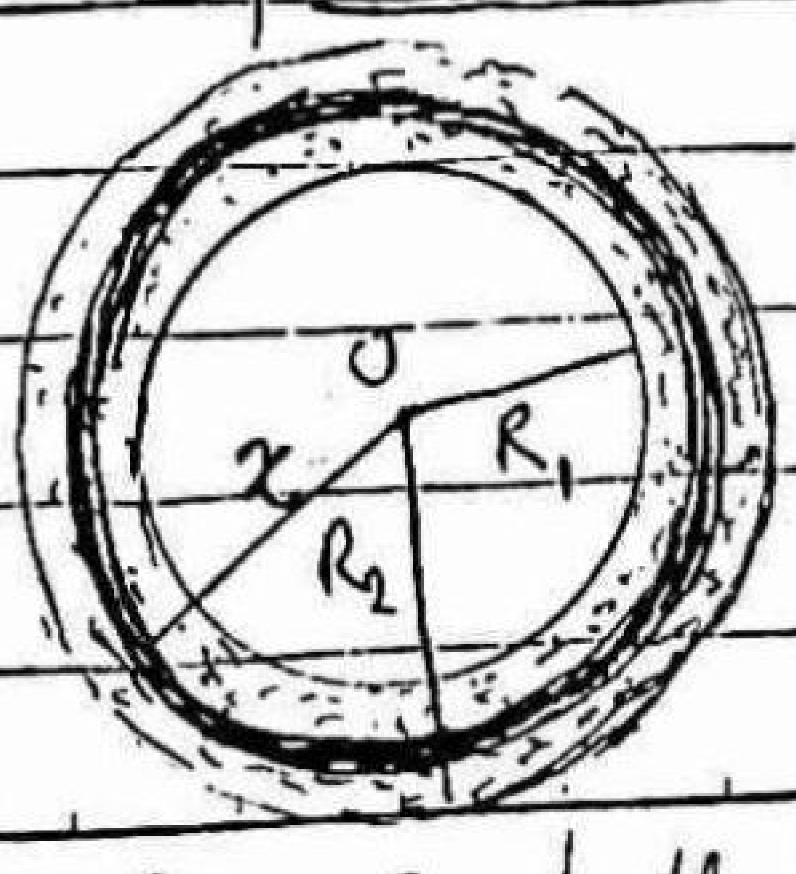
or, $$I_T = \frac{MR^2}{4} + MR^2$$

or, $$\boxed{I_T = \frac{5}{4} MR^2}$$

This gives moment of inertia about tangential axis.

6. Moment of inertia of an annular disc (Annulus)

Consider an annular disc of mass M centred at 'O' having internal radius R_1 and external radius R_2 such that area of



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annulus becomes (A) = $\pi R_2^2 - \pi R_1^2$

Mass per unit area = $\dfrac{M}{\pi(R_2^2 - R_1^2)}$

(a) About an axis through centre being perpendicular to the plane of annular ring.

Let's take an elementary ring of radius x and width $dx \to 0$ such that mass of elementary ring is given by;

$$(dm) = \frac{M}{\pi(R_2^2 - R_1^2)} \times 2\pi x\, dx$$

$$\text{or, } dm = \frac{2M}{(R_2^2 - R_1^2)} x\, dx$$

Moment of inertia of the elementary ring

$$dI = dm\, x^2$$

$$\text{or, } dI = \frac{2M}{(R_2^2 - R_1^2)} x^3\, dx \quad \text{--- (1)}$$

Now the moment of inertia of entire annular disc about axis through centre being perpendicular to its plane is given by;

$$I = \int_{R_1}^{R_2} \frac{2M}{(R_2^2 - R_1^2)} x^3\, dx$$

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$$I = \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{x^4}{4} \right]_{R_1}^{R_2}$$

$$\text{or } I = \frac{M}{2(R_2^2 - R_1^2)} \left(R_2^4 - R_1^4 \right)$$

$$\text{or } \boxed{I = \frac{1}{2} M \left(R_2^2 + R_1^2 \right)} \text{ —— } (2)$$

(b) About diametrical axis

$$\boxed{I_D = I/2 = \frac{1}{4} M (R_2^2 + R_1^2)}$$

(c) About tangential axis parallel to diametrical axis:

$$I_T = I_D + M R_2^2$$

$$\text{or } I_T = \frac{1}{4} M (R_2^2 + R_1^2) + M R_2^2$$

$$\text{or } \boxed{I_T = M \left[\frac{5R_2^2 + R_1^2}{4} \right]}$$

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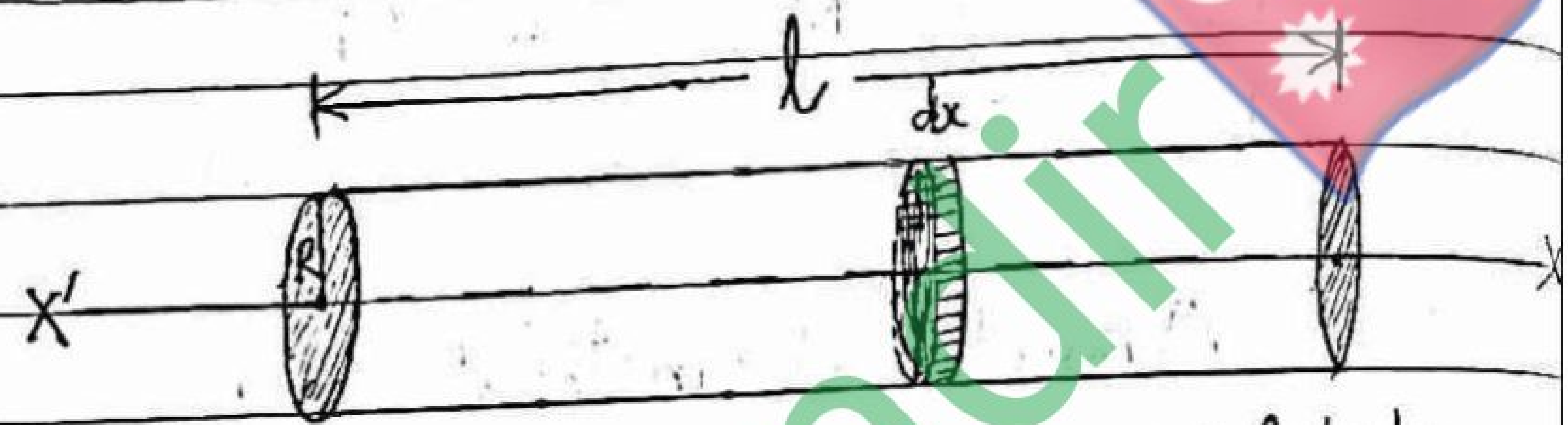
7 Moment of inertia of a solid cylinder

Consider a solid cylinder of mass M, radius R and length l such that its mass per unit length becomes M/l.

(a) About its own axis of symmetry



The solid cylinder can be supposed to be made up of large number of coaxial discs piled together. Let's take one such elementary disc of width $dx \to 0$ such that;

mass of the disc $(dm) = \frac{M}{l} dx$

So, moment of inertia of elementary disc is given by; $dI = \frac{1}{2} dm R^2$

or, $dI = \frac{1}{2} \frac{M}{l} R^2 dx$ —— ①

So, moment of inertia of the entire cylinder can be found by integrating equation ① within limits 0 to l, i.e.,

$$I = \int_0^l \frac{1}{2} \frac{M}{l} R^2 \, dx$$

or, $I = \frac{1}{2} \frac{MR^2}{l} \int_0^l dx = \frac{1}{2} \frac{MR^2}{l} \cdot l$

or, $\boxed{I = \frac{1}{2} MR^2}$ —— ②

This gives moment of inertia of solid cylinder about its own axis.

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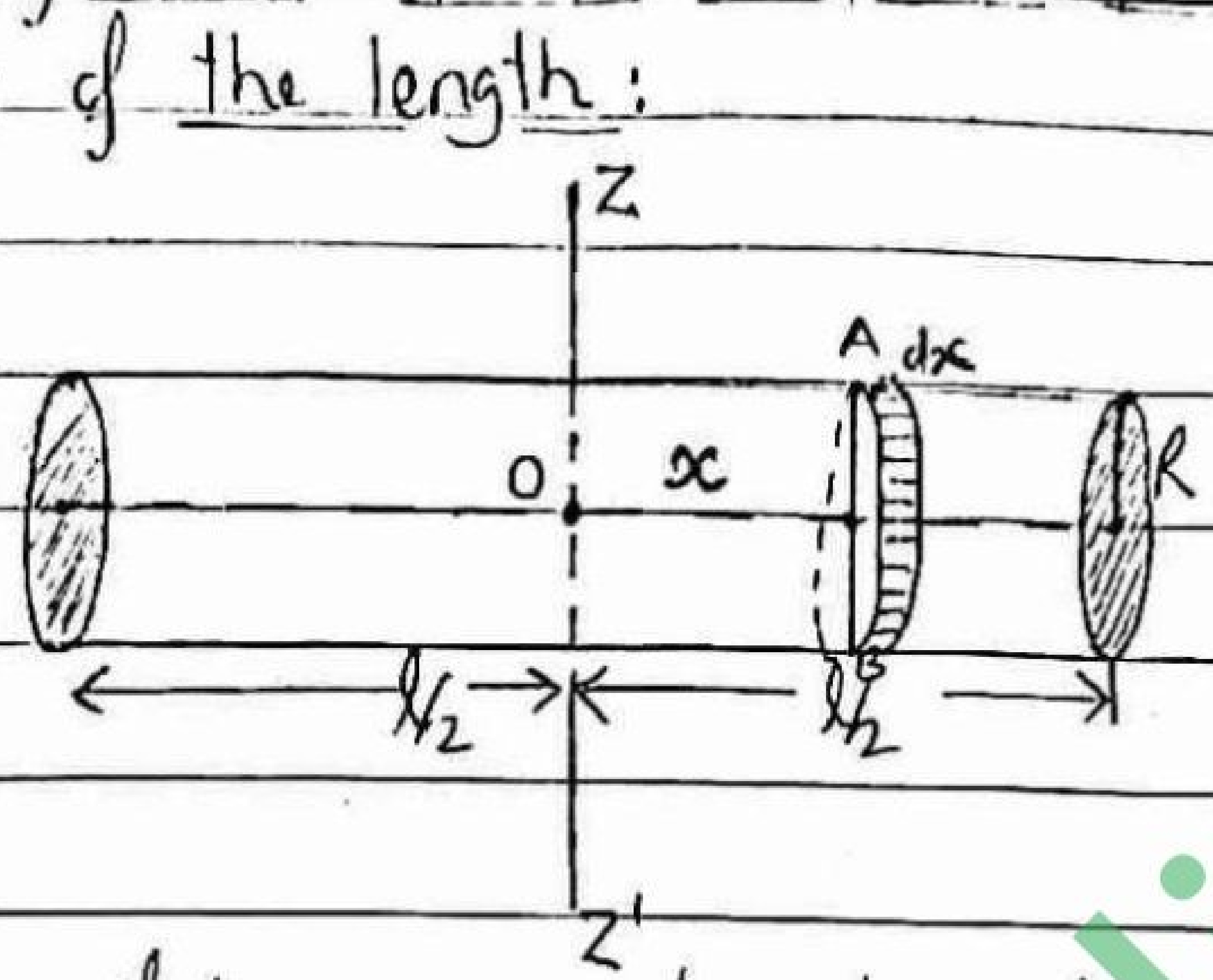
(b) Moment of inertia about an axis being perpendicular bisector of the length:



Let's take an elementary disc of width dx at distance x from given axis ZZ' so that

mass of elementary disc $(dm) = \frac{M}{l} dx$

moment of inertia of disc about diametrical axis AB is given by $dI' = \frac{1}{4}(dm) R^2$

or, $dI' = \frac{1}{4} \frac{M}{l} R^2 dx$

So, moment of inertia of the elementary disc about given axis ZZ', according to parallel axis theorem is given by

$$dI = dI' + dm\, x^2$$

$$\text{or, } dI = \frac{1}{4}\frac{M}{l}R^2 dx + \frac{M}{l}x^2 dx \quad \text{--- (1)}$$

Now, moment of inertia of entire solid cylinder about axis ZZ' can be found by integrating equation (1) within limits $x = 0$ to $x = l/2$ and multiplying by 2, i.e.,

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$$I = 2\int_0^{l/2}\left[\frac{1}{4}\frac{M}{l}R^2dx + \frac{M}{l}x^2dx\right]$$

or, $$I = 2\left[\int_0^{l/2}\frac{1}{4}\frac{M}{l}R^2dx + \int_0^{l/2}\frac{M}{l}x^2dx\right]$$

or, $$I = 2\left[\frac{1}{4}\frac{MR^2}{l}\left\{x\right\}_0^{l/2} + \frac{M}{l}\left\{\frac{x^3}{3}\right\}_0^{l/2}\right]$$

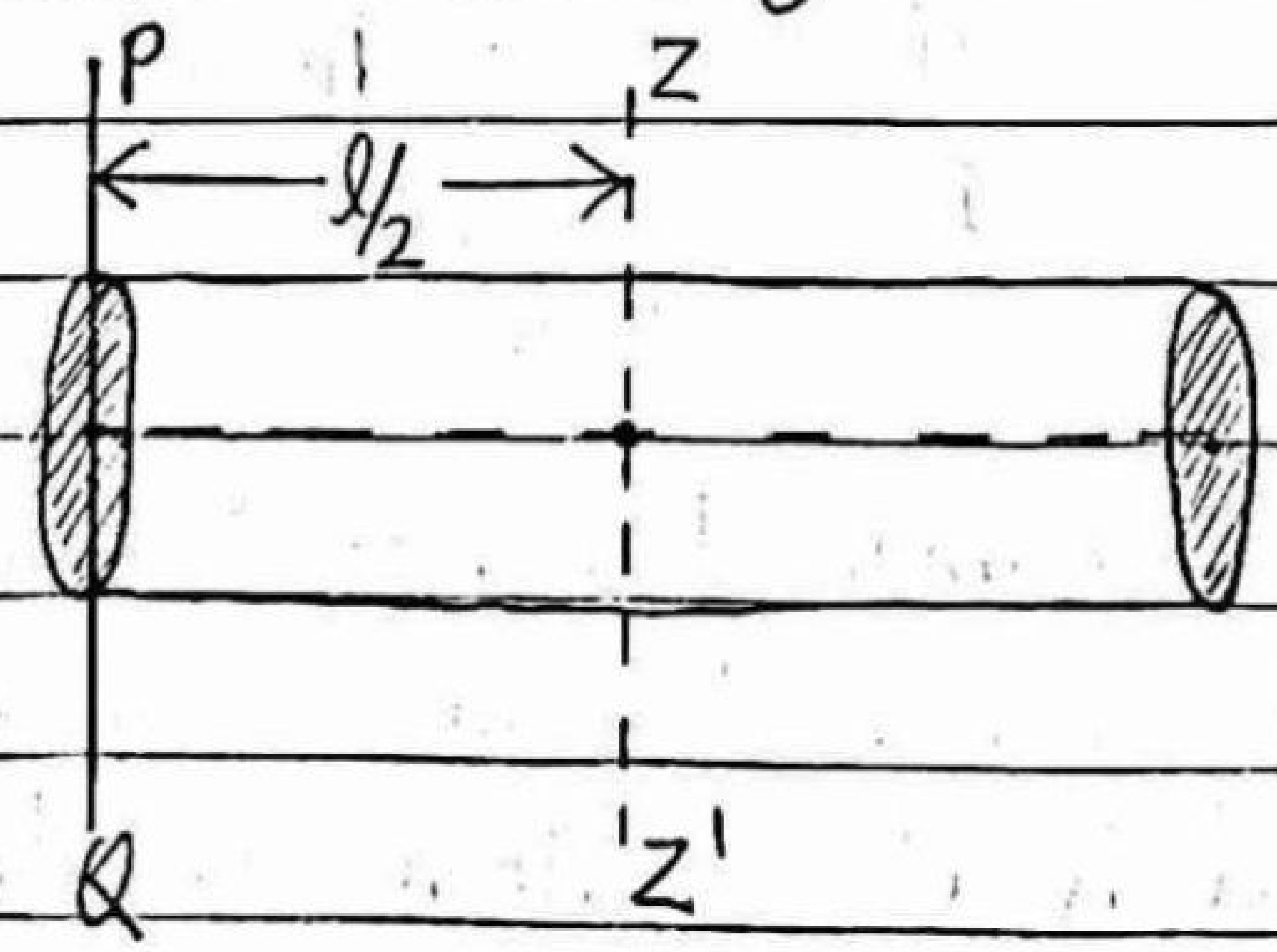
or, $$I = \frac{1}{2}\frac{MR^2}{l}\times\frac{l}{2} + \frac{2M}{3l}\frac{l^3}{8}$$

or, $$I = \frac{1}{4}MR^2 + \frac{Ml^2}{12}$$

or, $$\boxed{I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)} \quad \text{--- (2)}$$

This gives moment of inertia of solid cylinder about an axis that passes through centre being perpendicular to the length.

(c) About an axis from one end of cylinder being perpendicular to the length:



Date:

We know, moment of inertia of the solid cylinder about axis ZZ' is given by

$$I_Z = I_{CM} = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)$$

So, moment of inertia about axis from one end being perpendicular to the length, i.e. axis PQ according to parallel axes theorem is given by

$$I = I_{CM} + M(l/2)^2$$

or, $I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right) + \frac{Ml^2}{4}$

or, $I = M\left[\frac{R^2}{4} + \frac{l^2}{12} + \frac{l^2}{4}\right]$

or, $\boxed{I = M\left[\frac{R^2}{4} + \frac{l^2}{3}\right]}$

8. Moment of inertia of a hollow thin cylinder

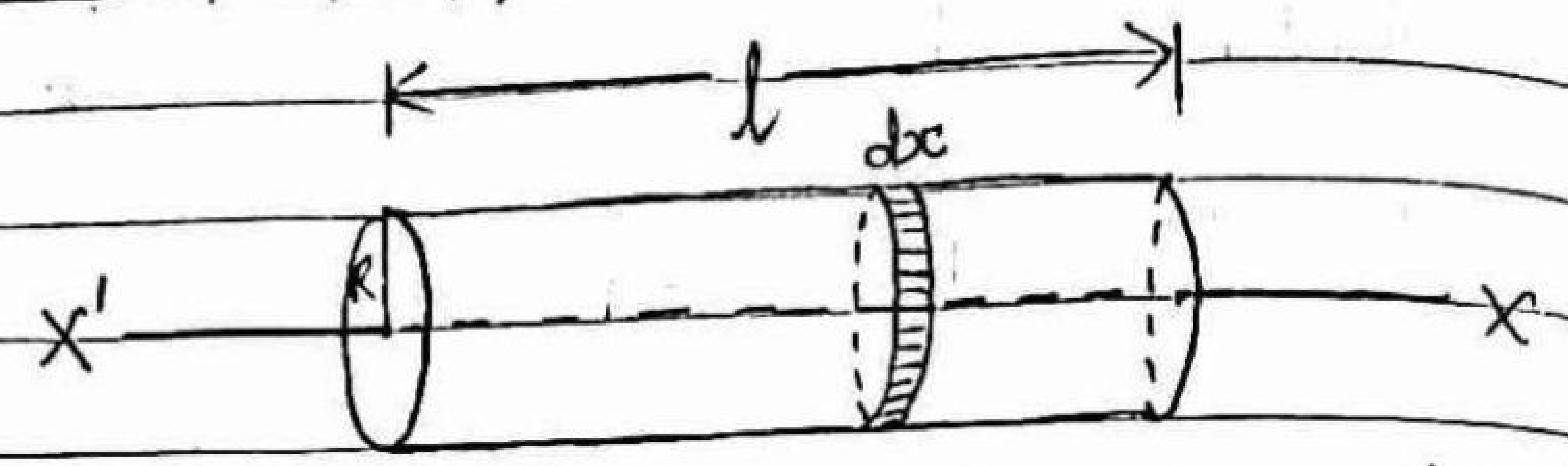
Consider a hollow and thin cylinder of mass M and radius R having length l such that mass per unit length of the cylinder becomes $\frac{M}{l}$.

(a) About its own axis of symmetry:

The axis of symmetry passes through inside being perpendicular to circular cross-

Date:

section as shown;



The hollow cylinder can be supposed to be made up of coaxial rings piled together. Let's take one such elementary ring of width $dx \to 0$ such that,

mass of elementary ring $(dm) = \frac{M}{l} dx$

moment of inertia about axis of symmetry

$$dI = dm\, R^2$$

or, $$dI = \left(\frac{M}{l} dx\right) R^2$$

or, $$dI = \frac{M}{l} R^2 \, dx \quad \text{———} \quad (1)$$

Now, the moment of inertia of entire hollow cylinder can be obtained as,

$$I = \int_0^l \frac{M}{l} R^2 \, dx$$

or, $$I = \frac{M}{l} R^2 \left[x\right]_0^l$$

or, $$\boxed{I = MR^2} \quad \text{———} \quad (2)$$

This gives Moment of inertia of a thin hollow cylinder about its own axis of symmetry.

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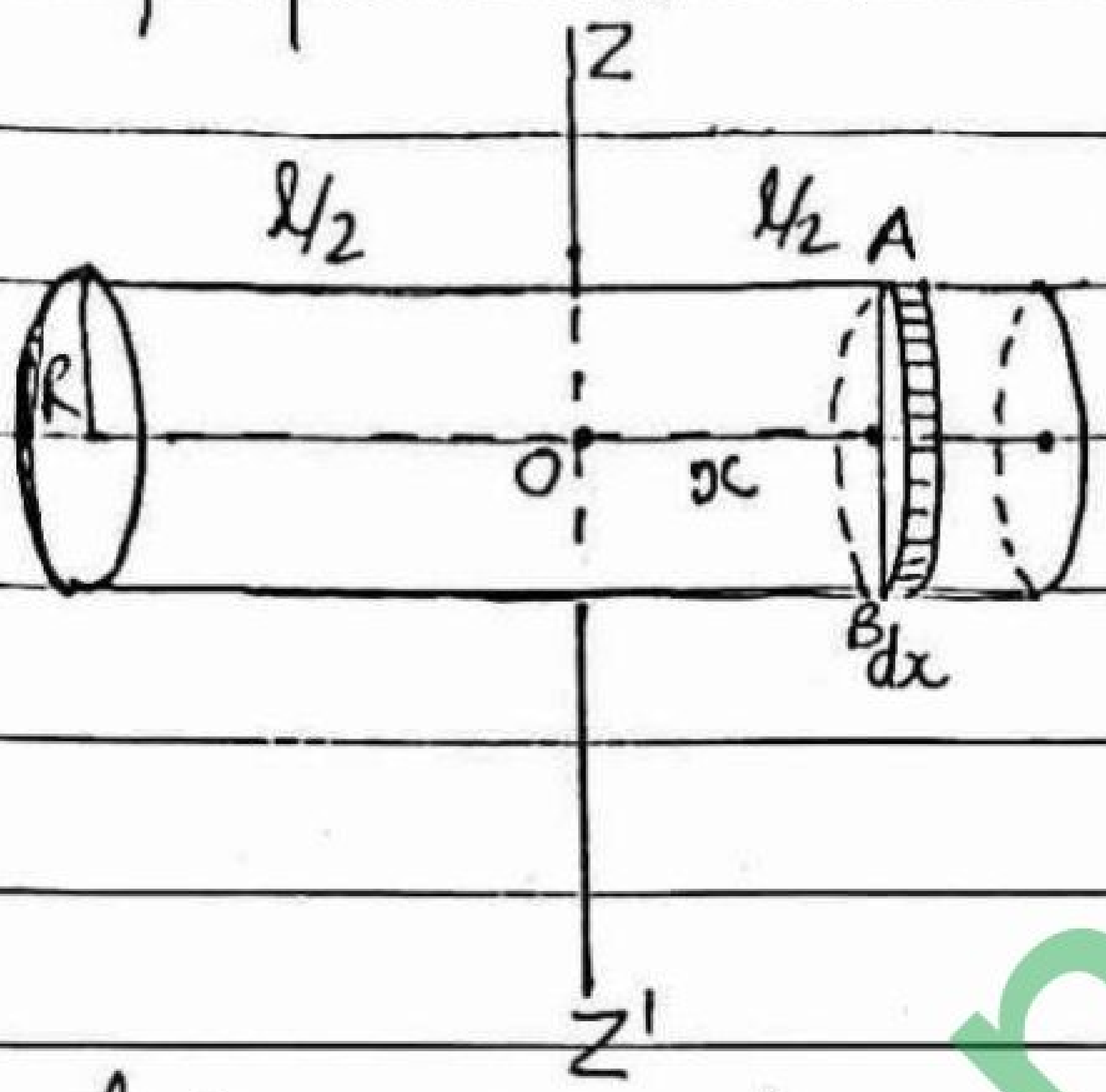
(b) About an axis that passes through centre being perpendicular to the length:



Let's take an elementary ring of width $dx \to 0$ at distance x from given axis ZZ' such that

$$\text{mass of ring } (dm) = \frac{M}{l} dx$$

moment of inertia about diametrical axis AB parallel to ZZ' is

$$dI' = \frac{1}{2}(dm) R^2$$

or, $$dI' = \frac{1}{2}\left(\frac{M}{l} dx\right) R^2$$

or, $$dI' = \frac{1}{2} \frac{MR^2}{l} dx$$

So, moment of inertia about given axis ZZ', according to parallel axes theorem

$$dI = dI' + (dm)x^2$$

or, $$dI = \frac{1}{2} \frac{MR^2}{l} dx + \frac{M}{l} x^2 dx \quad \text{--- (1)}$$

Date:

Now, moment of inertia of entire hollow thin cylinder about axis ZZ' can be found by integrating equation ① within limits 0 to $l/2$ and multiplying it by 2, i.e.,

$$I = 2\int_0^{l/2}\left[\frac{1}{2}\frac{MR^2}{l}dx + \frac{M}{l}x^2dx\right]$$

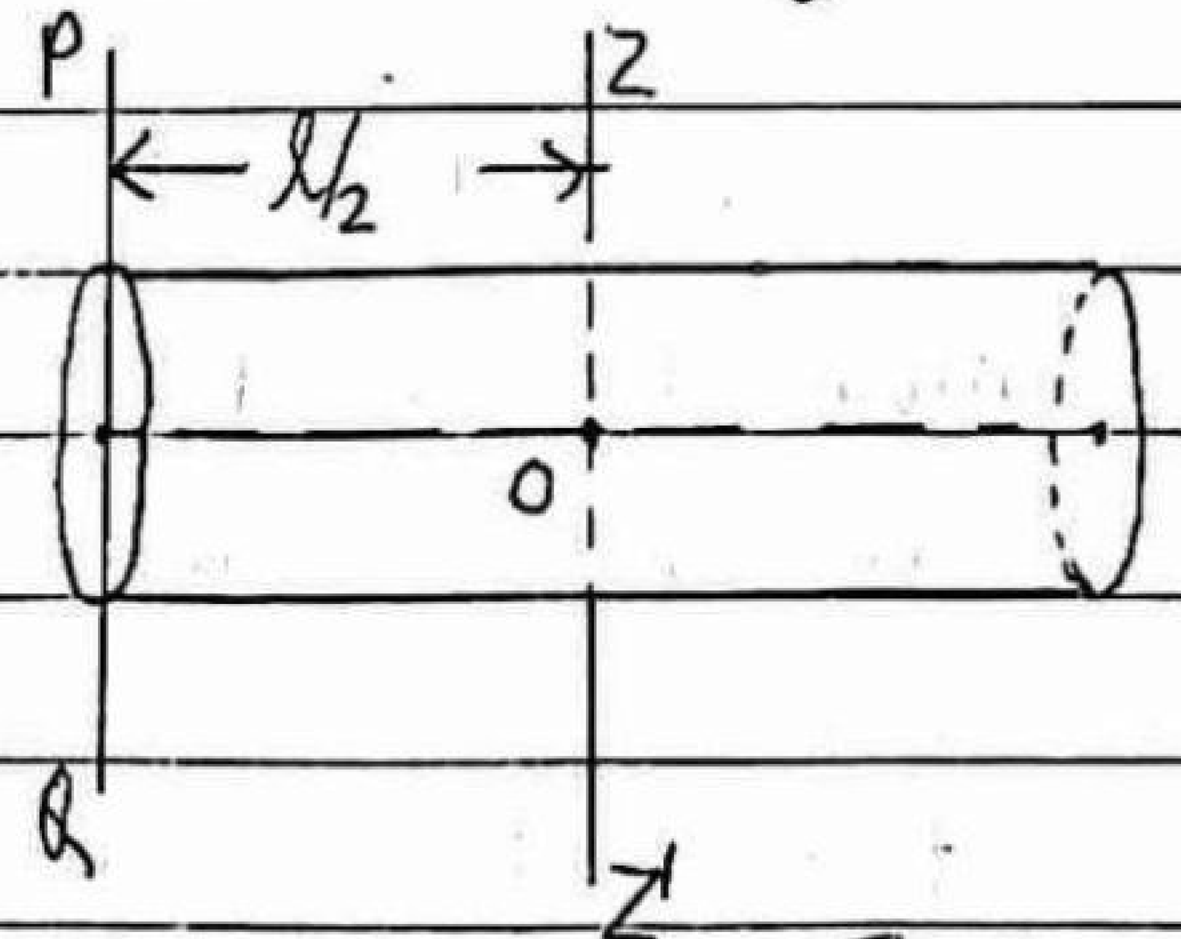
or $$I = \frac{MR^2}{l}\int_0^{l/2}dx + \frac{2M}{l}\int_0^{l/2}x^2dx$$

or $$I = \frac{MR^2}{l}\times l/2 + \frac{2M}{l}\left[\frac{x^3}{3}\right]^{l/2}$$

or $$I = \frac{1}{2}MR^2 + \frac{Ml^2}{12}$$

or $$I = M\left[\frac{R^2}{2} + \frac{l^2}{12}\right] \quad \text{——— ②}$$

(c) About an axis that passes from one end being perpendicular to length:



We have $I_Z = I_{cm} = M\left[\frac{R^2}{2} + \frac{l^2}{12}\right]$

Date:

So, moment of inertia of the cylinder about axis PQ, according to parallel axes theorem becomes

$$I = I_{cm} + M(l/2)^2$$

or, $$I = M\left[\frac{R^2}{2} + \frac{l^2}{12}\right] + \frac{Ml^2}{4}$$

or, $$I = M\left[\frac{R^2}{2} + \frac{l^2}{12} + \frac{l^2}{4}\right]$$

or, $$\boxed{I = M\left[\frac{R^2}{2} + \frac{l^2}{3}\right]}$$

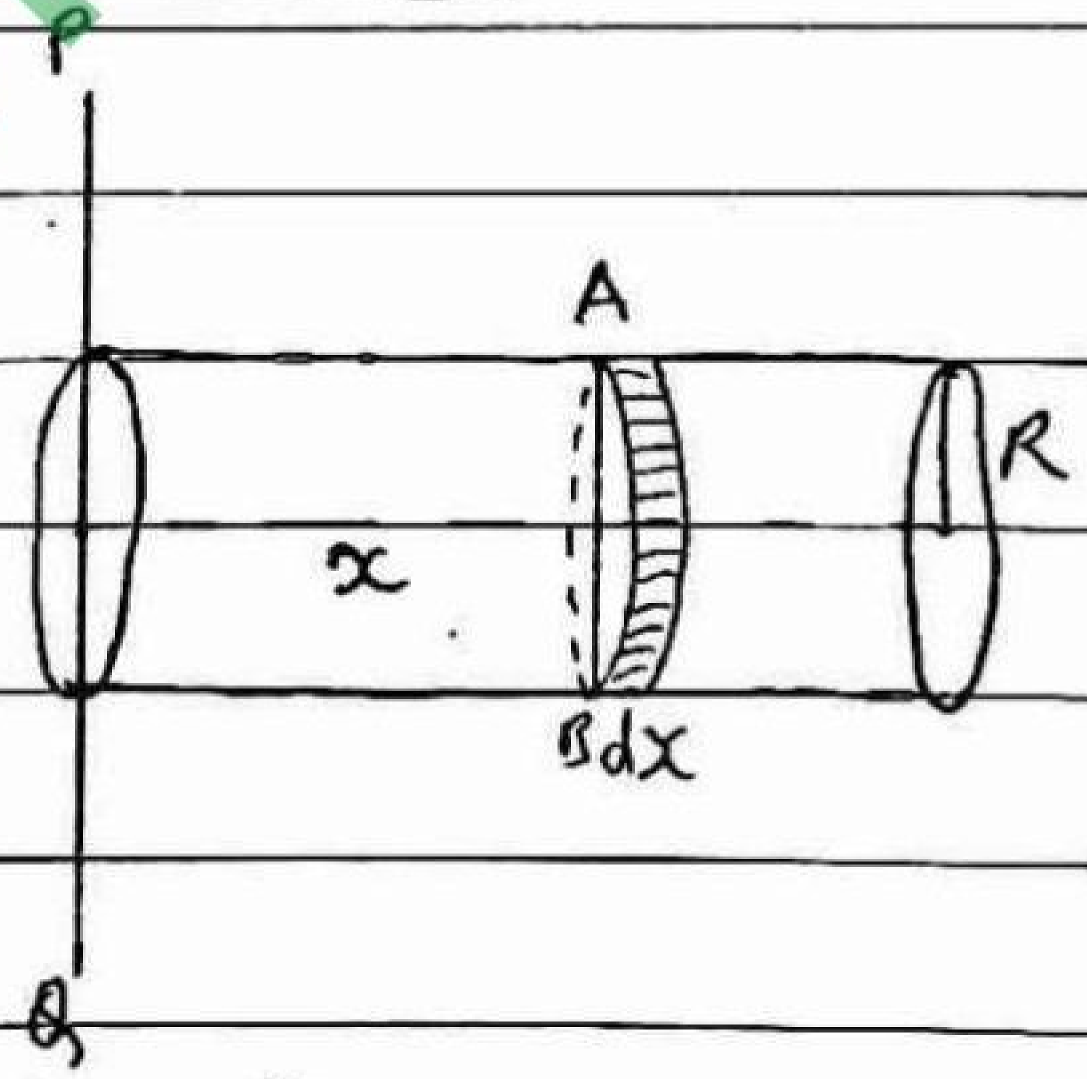
Alternative method

mass of elementary ring $(dm) = \frac{M}{l}dx$

M.I. about diametrical axis AB

$$(dI') = \frac{1}{2}dmR^2$$

or, $$dI' = \frac{1}{2}\frac{M}{l}R^2dx$$



M.I. about PQ $(dI) = dI' + dm\,x^2$

or $$dI = \frac{1}{2}\frac{M}{l}R^2dx + \frac{M}{l}x^2dx$$

Total M.I. $$I = \int_0^l \left[\frac{1}{2}\frac{MR^2}{l}dx + \frac{M}{l}x^2dx\right]$$

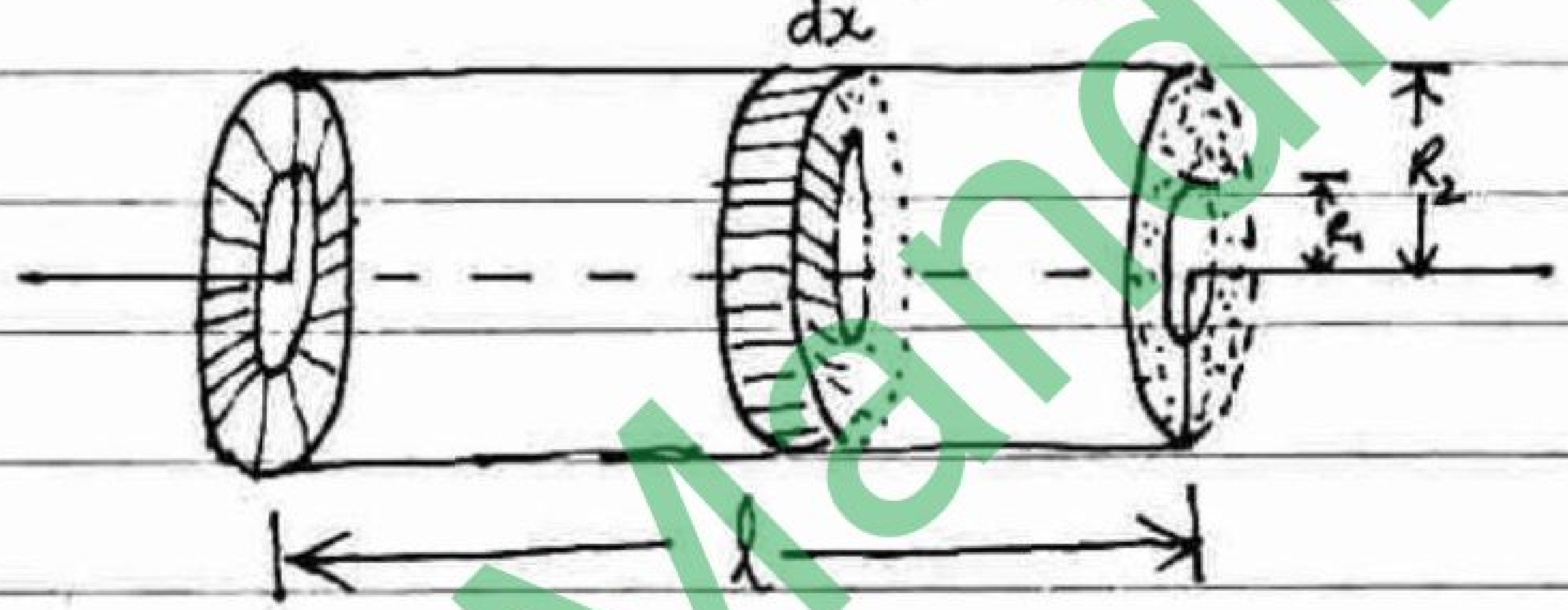
or, $$I = \frac{1}{2}\frac{MR^2}{l}\times l + \frac{M}{l}\frac{l^3}{3}$$

or, $$I = M\left[\frac{R^2}{2} + \frac{l^2}{3}\right]$$

10. Moment of inertia of a thick hollow cylinder

Suppose a hollow cylinder of mass M with inner radius R_1 and outer radius R_2 and length l such that mass per unit length becomes $\frac{M}{l}$.

(a) About its own axis of symmetry:



The hollow cylinder can be supposed to be made up of large number of annular discs. Let's take one such elementary disc (annulus) having width dx such that;

mass of elementary annular disc

$$dm = \frac{M}{l} dx$$

moment of inertia of elementary annular disc about axis of symmetry

$$dI = \left(\frac{M}{l} dx\right)\left(\frac{R_1^2 + R_2^2}{2}\right)$$

So, moment of inertia of entire thick hollow cylinder about its own axis of symmetry is given by

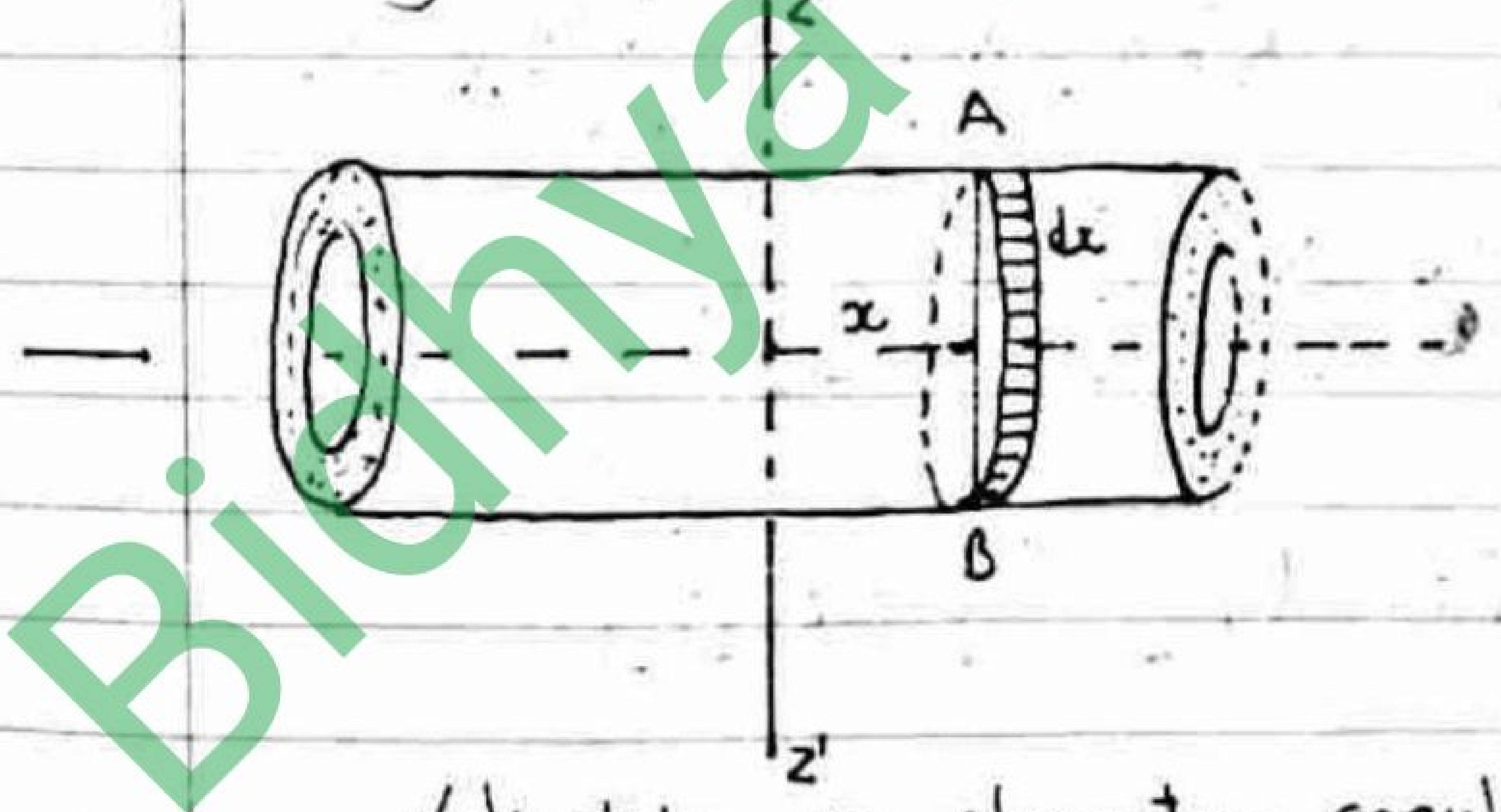
$$I = \int_0^l \left(\frac{M}{l}dx\right)\left(\frac{R_1^2+R_2^2}{2}\right)$$

or, $$I = \frac{M}{l}\left(\frac{R_1^2+R_2^2}{2}\right)\int_0^l dx$$

or, $$I = \frac{M}{l}\left(\frac{R_1^2+R_2^2}{2}\right)\times l$$

or, $$\boxed{I = M\left(\frac{R_1^2+R_2^2}{2}\right)}$$

(b) About an axis that passes through centre being perpendicular to the length:



Let's take an elementary annular disc of width dx at distance x from axis ZZ'. So,

mass of elementary annular disc $(dm) = \frac{M}{l}dx$

moment of inertia about diameter AB $(dI') = \frac{1}{4}dm\left(R_1^2+R_2^2\right)$

or, $dI' = \frac{1}{4}\frac{M}{l}\left(R_1^2+R_2^2\right)dx$

Date:

moment of inertia about ZZ' $(dI) = dI' + dm\, x^2$

$$\text{or, } dI = \frac{1}{4}\frac{M}{l}(R_1^2+R_2^2)dx + \frac{M}{l}x^2 dx \quad \text{--- (1)}$$

Now, moment of inertia of entire thick hollow cylinder can be found by integrating equation (1) within limits $x=0$ to $x = l/2$ and multiplying by 2 i.e.

$$I = 2\int_0^{l/2}\left[\frac{1}{4}\frac{M}{l}(R_1^2+R_2^2)dx + \frac{M}{l}x^2 dx\right]$$

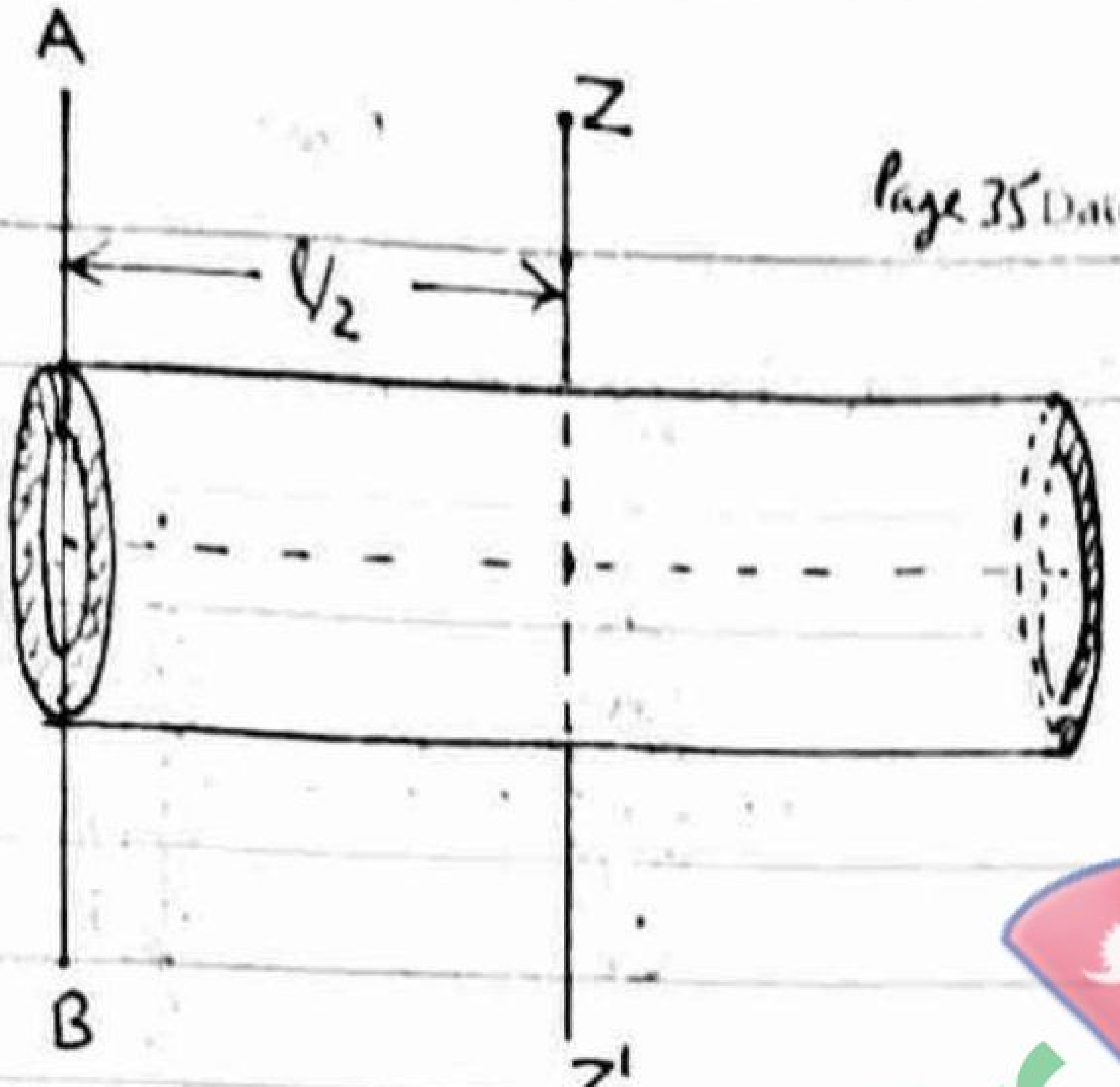
$$\text{or, } I = 2\left[\frac{1}{4}\frac{M}{l}(R_1^2+R_2^2)\int_0^{l/2}dx + \frac{M}{l}\int_0^{l/2}x^2 dx\right]$$

$$\text{or, } I = \frac{1}{2}\frac{M}{l}(R_1^2+R_2^2)\times\frac{l}{2} + \frac{2M}{3l}\times\frac{l^3}{8}$$

$$\text{or, } I = M\left(\frac{R_1^2+R_2^2}{4}\right) + \frac{Ml^2}{12}$$

$$\text{or, } I = M\left[\frac{R_1^2+R_2^2}{4} + \frac{l^2}{12}\right]$$

(c) About an axis from one end being perpendicular to the length:



We have $I_{cm} = I_{ZZ'} = M\left[\frac{R_1^2+R_2^2}{4} + \frac{l^2}{12}\right]$

So, from parallel axes theorem

$$I_{AB} = I_{cm} + M\left(\frac{l}{2}\right)^2$$

or, $I_{AB} = M\left[\frac{R_1^2+R_2^2}{4} + \frac{l^2}{12}\right] + M\,\frac{l^2}{4}$

or, $I_{AB} = M\left[\frac{R_1^2+R_2^2}{4} + \frac{l^2}{12} + \frac{l^2}{4}\right]$

or, $\boxed{I_{AB} = M\left[\frac{R_1^2+R_2^2}{4} + \frac{l^2}{3}\right]}$

* Alternative method:

Let's take an elementary annular disc of width $dx \to 0$ at distance x from the axis AB from one end of the thick hollow cylinder

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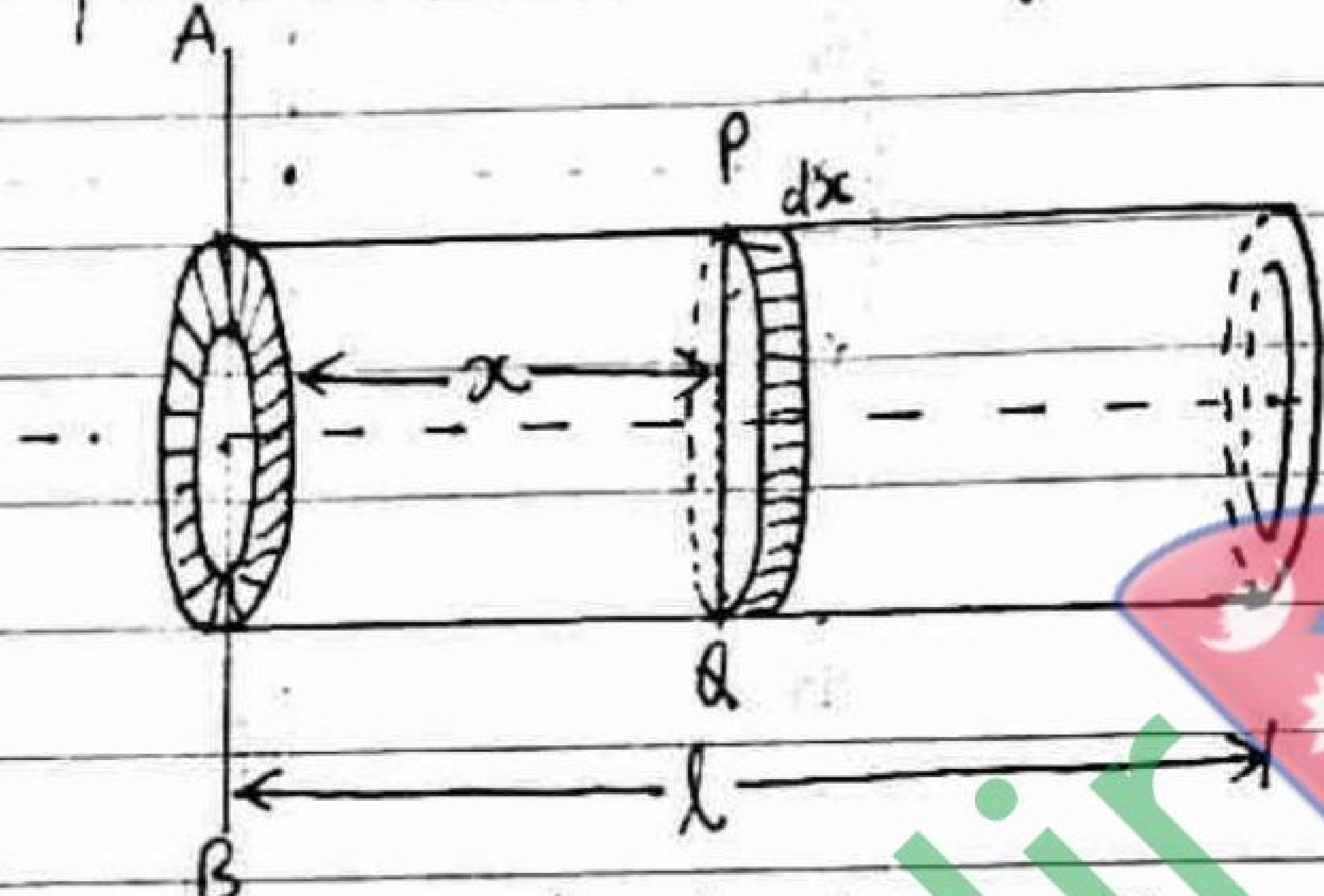
being perpendicular to the length such that



Mass of elementary annular disc $(dm) = \frac{M}{l} dx$

Moment of inertia about diameter PQ $(dI') = \frac{1}{4} dm (R_1^2 + R_2^2)$

or $dI' = \frac{1}{4} \frac{M}{l} (R_1^2 + R_2^2) dx$

Moment of inertia about the axis AB through one end

$$dI = dI' + dm\, x^2$$

or $dI = \frac{1}{4} \frac{M}{l} (R_1^2 + R_2^2) dx + \frac{M}{l} x^2 dx$ — (1)

Now the moment of inertia of entire thick hollow cylinder can be found by integrating equation (1) within limits $x=0$ to $x=l$

i.e.,

$$I = \frac{M}{l} \left(\frac{R_1^2 + R_2^2}{4}\right) \int_0^l dx + \frac{M}{l} \int_0^l x^2 dx$$

or $I = M\left(\frac{R_1^2 + R_2^2}{4}\right) + \frac{Ml^2}{3} = M\left[\frac{R_1^2 + R_2^2}{4} + \frac{l^2}{3}\right]$

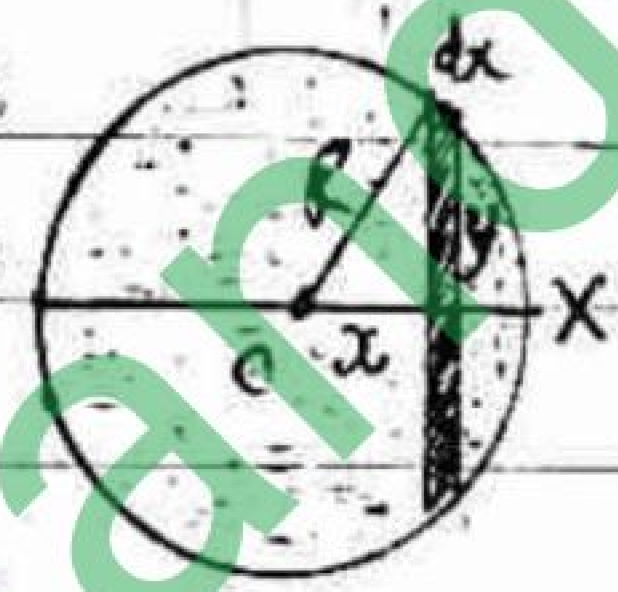
Date:

Moment of inertia of a solid sphere: (About diameter)

Consider a solid sphere of mass M, radius R centred at 'O' having uniform density ρ such that

$$M = \frac{4}{3}\pi R^3 \rho$$

To find moment of inertia of the solid sphere about diameter XX', let's take an elementary circular disc of width dx



at distance x from the centre of the sphere such that the given axis XX' passes being perpendicular to the plane of the disc. Here,

$$\text{radius of the disc } (y) = \sqrt{R^2 - x^2}$$

$$\text{volume of the disc} = \pi y^2 dx = \pi (R^2 - x^2) dx$$

$$\text{mass of the disc } (dm) = \pi (R^2 - x^2) \rho \, dx$$

$$\text{moment of inertia of the disc } (dI) = \frac{1}{2} (dm) y^2$$

or, $$dI = \frac{1}{2} \pi (R^2 - x^2) \rho \, dx \, (R^2 - x^2)$$

or, $$dI = \frac{1}{2} \pi (R^2 - x^2)^2 \rho \, dx$$

or, $$dI = \frac{1}{2} \pi \rho \left(R^4 - 2R^2 x^2 + x^4 \right) dx \quad \text{——} \ (1)$$

To find moment of inertia of entire solid sphere, equation ① should be integrated from limits $x=0$ to $x=R$ and multiplied by 2, i.e.,

$$I = 2\int_0^R \frac{1}{2}\pi\rho\left(R^4 - 2R^2x^2 + x^4\right)dx$$

or, $$I = \pi\rho\left[R^4\int_0^R dx - 2R^2\int_0^R x^2\,dx + \int_0^R x^4\,dx\right]$$

or, $$I = \pi\rho\left[R^4 \times R - 2R^2\frac{R^3}{3} + \frac{R^5}{5}\right]$$

or, $$I = \pi\rho\left[\frac{15R^5 - 10R^5 + 3R^5}{15}\right]$$

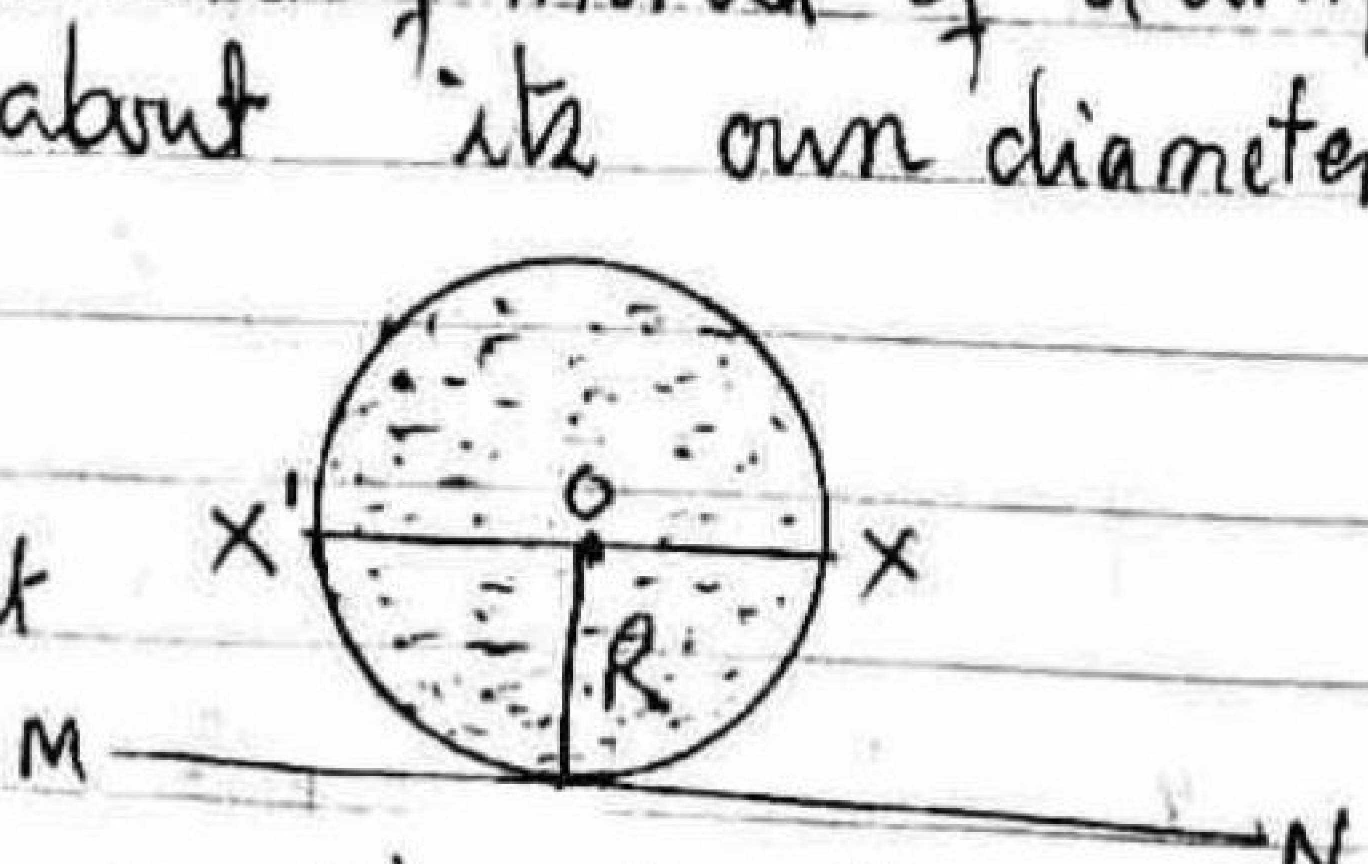
or, $$I = \frac{8}{15}\pi\rho R^5$$

or, $$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$$

or, $$\boxed{I = \frac{2}{5}MR^2} \quad \text{——— ②}$$

This gives moment of inertia of a uniform solid sphere about its own diameter.

About tangent :



If I_T be the M.I about tangent MN then

$I_T = I_D + MR^2$; where $I_D = I_{CM}$

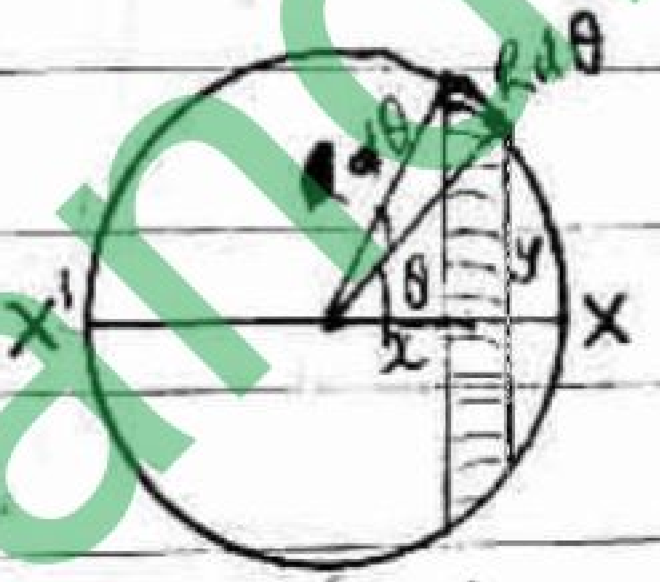
or $$I_T = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

Date:

Moment of inertia of a thin hollow sphere (spherical shell)
(About diameter)

Consider a spherical shell of mass M, radius R centred at 'O' with surface density σ such that

$$\sigma = \frac{M}{4\pi R^2}$$



To find moment of inertia of hollow sphere about diameter XX', let's first take an elementary ring of radius y at distance x from the centre of the shell and let the width of ring be $dx = Rd\theta$. Now, from geometry;

$$\text{radius of ring } (y) = R\sin\theta$$
$$\text{circumference} = 2\pi y = 2\pi R\sin\theta$$
$$\text{area} = (2\pi R\sin\theta)\, Rd\theta = 2\pi R^2 \sin\theta\, d\theta$$
$$\text{Mass } (dm) = 2\pi R^2 \sin\theta\, d\theta \cdot \sigma = 2\pi R^2 \sigma \sin\theta\, d\theta$$

$$\text{moment of inertia } (dI) = (dm)\, y^2$$
$$\text{or, } dI = (2\pi R^2 \sigma \sin\theta\, d\theta)\, R^2 \sin^2\theta$$
$$\text{or, } dI = 2\pi R^4 \sigma \sin^3\theta\, d\theta$$
$$\text{or, } dI = 2\pi R^4 \frac{M}{4\pi R^2} \sin^3\theta\, d\theta$$

or,

Date:

$$dI = \frac{1}{2} MR^2 \sin^3\theta \, d\theta \quad \text{———} \; (1)$$

The moment of inertia of entire spherical shell can be found by integrating equation (1) within limits $\theta = 0$ to $\theta = \pi$, i.e.,

$$I = \frac{1}{2}MR^2 \int_0^{\pi} \sin^3\theta \, d\theta$$

or, $$I = \frac{1}{2}.MR^2 \int_0^{\pi} \sin^2\theta . \sin\theta \, d\theta$$

or, $$I = \frac{1}{2}MR^2 \int_0^{\pi} (1 - \cos^2\theta) . \sin\theta \, d\theta$$

Let $\cos\theta = Z$ then $-\sin\theta \, d\theta = dZ$

or $\sin\theta \, d\theta = -dZ$

Also, for $\theta = 0$; $Z = 1$

for $\theta = \pi$; $Z = -1$

$$\therefore I = \frac{1}{2}MR^2 \int_1^{-1} (1 - Z^2)(-dZ)$$

or, $$I = \frac{1}{2}MR^2 \int_{-1}^{1} (1 - Z^2) \, dZ$$

or, $$I = \frac{1}{2}MR^2 \left[\int_{-1}^{1} dZ - \int_{-1}^{1} Z^2 dZ \right]$$

or, $$I = \frac{1}{2}MR^2 \left[Z\Big|_{-1}^{1} - \frac{Z^3}{3}\Big|_{-1}^{1} \right]$$

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or, $I = \frac{1}{2} MR^2 \left[\{1-(-1)\} - \left\{ \frac{1^3}{3} - \frac{(-1^3)}{3} \right\} \right]$

or, $I = \frac{1}{2} MR^2 \left[2 - \left\{ \frac{1}{3} + \frac{1}{3} \right\} \right]$

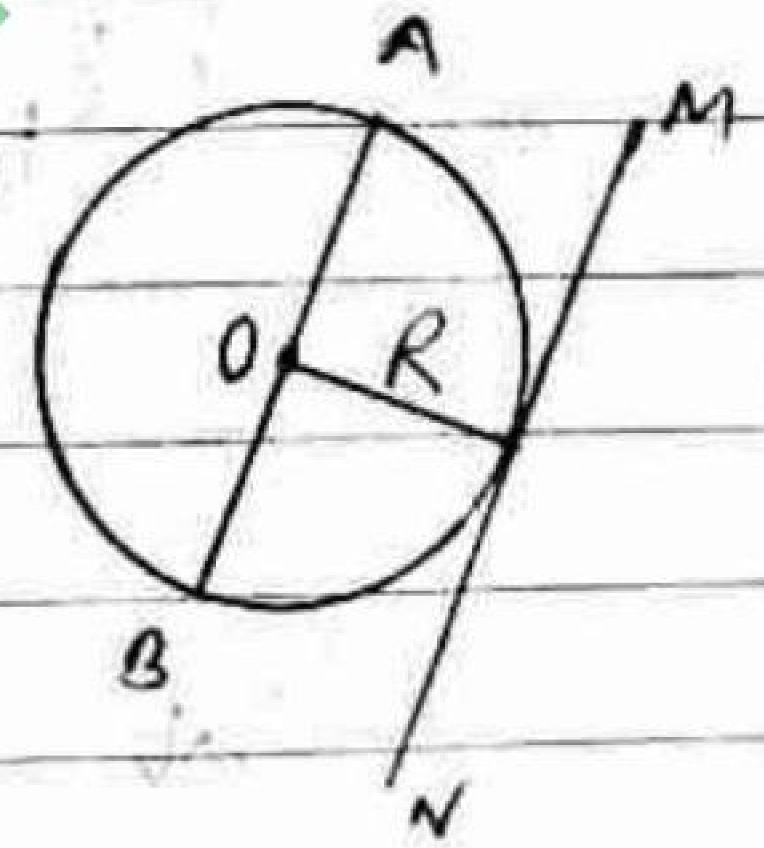
or, $I = \frac{1}{2} MR^2 \left[2 - \frac{2}{3} \right]$

or, $I = \frac{1}{2} MR^2 \times \frac{4}{3}$

or, $\boxed{I = \frac{2}{3} MR^2}$ ——— (2)

This gives moment of inertia of a spherical shell or thin hollow sphere.

About tangent.



If I_D be the moment of inertia of hollow sphere about diameter AB then

$$I_D = I_{cm}$$

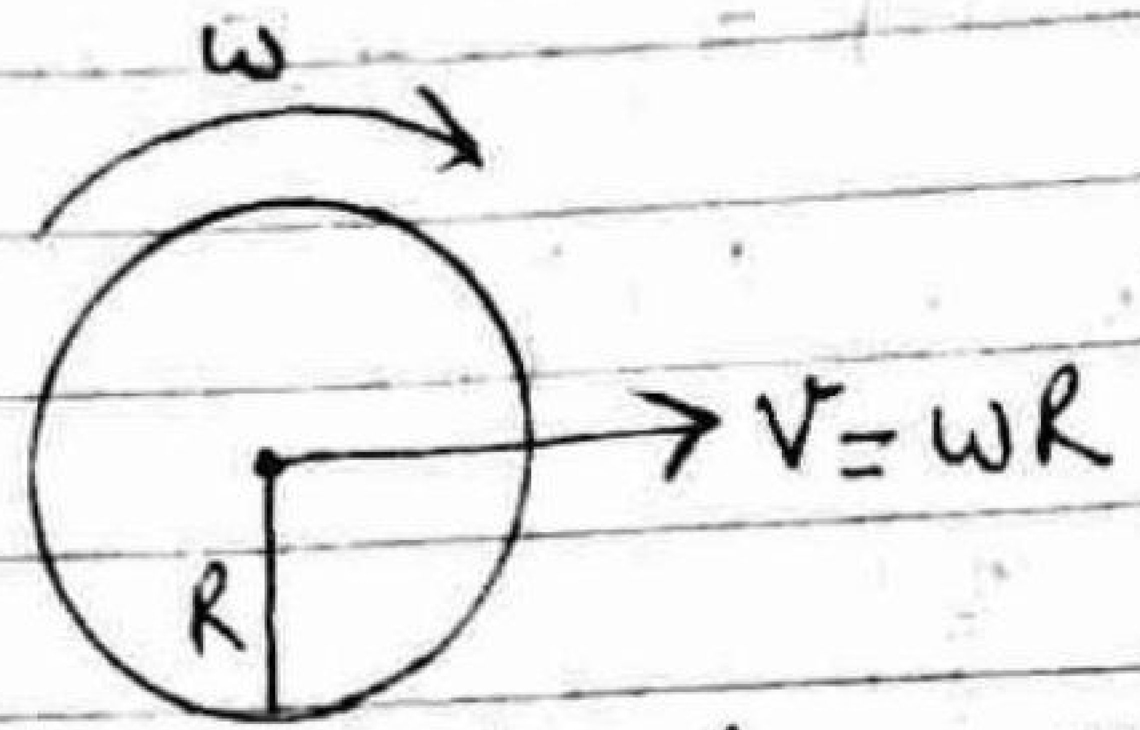
So, moment of inertia about tangent MN, parallel to diameter AB, according to parallel axes theorem is

$$I_T = I_{cm} + MR^2$$

or, $I_T = I_D + MR^2 = \frac{2}{3} MR^2 + MR^2 = \frac{5}{3} MR^2$

Date:

Kinetic energy of rolling body



If a body of mass M, radius R, moment of inertia I is rolling with angular velocity ω without slipping then linear velocity of the centre of mass will be

$$V = \omega R \Rightarrow \omega = V/R$$

So, total kinetic energy of the rolling body

$$K.E_{roll} = K.E_{trans} + K.E_{rot} \quad \text{--- (1)}$$

where, $K.E_{trans} = \frac{1}{2} MV^2$ (translational kinetic energy)

$K.E_{rot} = \frac{1}{2} I\omega^2$ (rotational kinetic energy)

Equation (1) becomes;

$$K.E_{roll} = \frac{1}{2} MV^2 + \frac{1}{2} I\omega^2$$

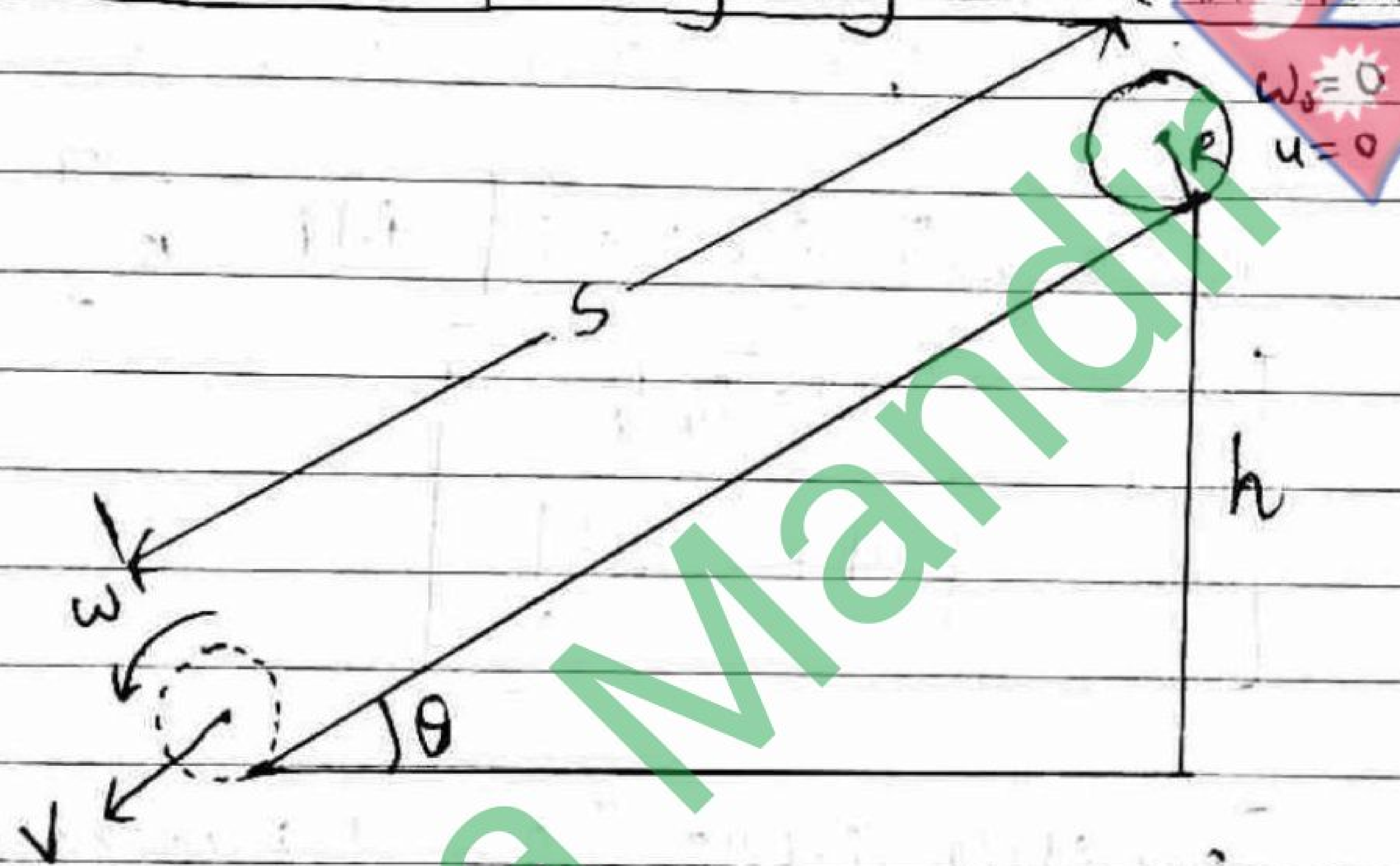
or, $$K.E_{roll} = \frac{1}{2} MV^2 + \frac{1}{2} I \, V^2/R^2$$

Date:

or $$\boxed{K.E_{roll} = \frac{1}{2} v^2 \left[M + I/R^2 \right]} \quad \text{———} \ (2)$$

This gives kinetic energy of rolling body.

Acceleration of rolling body down an incline



Let a body of mass M, radius R and moment of inertia I starts to roll from the top of an incline of length S, height h that makes angle θ with the horizontal. The body finally achieves linear velocity v and angular velocity ω at the bottom of the incline.

From the principle of conservation of energy

P.E at top = Rolling K.E at bottom

$$Mgh = \frac{1}{2}V^2\left[M + I/R^2\right]$$

or, $$Mg\,s\,\sin\theta = \frac{1}{2}V^2\left[M + I/R^2\right] \quad \because h = s\sin\theta$$

Also, $$V^2 = u^2 + 2as = 2as \quad [\because u = 0]$$

or, $$Mg\,s\,\sin\theta = \frac{1}{\cancel{2}} \times \cancel{2}as\left[M + I/R^2\right]$$

or, $$\boxed{a = \frac{Mg\sin\theta}{\left[M + I/R^2\right]}}$$

This gives acceleration of a rolling body down an incline.

* For Ring $I = MR^2 \Rightarrow a = \frac{1}{2}g\sin\theta$
* For disc $I = \frac{1}{2}MR^2 \Rightarrow a = \frac{2}{3}g\sin\theta$
* For thin cylinder $I = MR^2 \Rightarrow a = \frac{1}{2}g\sin\theta$
* For solid cylinder $I = \frac{1}{2}MR^2 \Rightarrow a = \frac{2}{3}g\sin\theta$
* For spherical shell $I = 2/3\,MR^2 \Rightarrow a = \frac{3}{5}g\sin\theta$
* For solid sphere $I = \frac{2}{5}MR^2 \Rightarrow a = 5/7\,g\sin\theta$