Page - 1

Chapter - 17: Elementary vector Analysis

* **Vectors** :- Vector is a physical quantity that can be completely described by considering both magnitude and direction. A vector is represented by arrow (→) headed letters like $\vec{F}$ for force.

If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ be a vector then it's magnitude is obtained by relation

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

* **Types of vector** :- The vector quantities are of following two types :-

(i) **polar vector** :- The vector which describes the translational motion is called polar vector. for eg. i velocity, momentum, force etc.

(ii) **Axial vector** :- The vector which describes the rotational motion is called axial vector.

* **Product of vectors** :- There is possibility of two types of product between vectors. They are

(i) **Dot product** : The dot product between two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\cdot\vec{B}$ and it is defined by $\vec{A}\cdot\vec{B} = AB\cos\theta$ where θ = angle between directions of $\vec{A}$ and $\vec{B}$.

(ii) **Cross product** : The cross product between two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$ and it is defined by $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ $\quad 0 \leq \theta \leq \pi$

where $\hat{n}$ is a unit vector representing the direction of cross product.

* **Addition of vectors**: The sum of two vectors is called resultant vector. So if $\vec{A}+\vec{B}=\vec{R}$ then magnitude and direction of $\vec{R}$ will be

$$R=\sqrt{A^2+2AB\cos\theta+B^2}$$

$$\alpha=\tan^{-1}\left(\frac{B\sin\theta}{A+B\cos\theta}\right)$$

Here θ is the angle between vectors.

* **Scalar point function**:- Scalar point fontion is a function that describes the physical phenomena in terms of scalar quantity. for example: temperature distribution around the heated body, electric potential distribution around the charge etc.

* **Vector point function**:- It is a function that describes the physical phenomena in terms of vector quantity. for example:- The electric field, intensity distribution around any charge, the velocity distribution of flowing liquid.

* **Vector differtial operation (del)**:- The vector differential operator is denoted by ∇ and it is defined by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

When this operator operates a scalar point function, it gives the vector result.

* Gradient :- The gradient of a scalar function ϕ is denoted by grad ϕ or $\nabla\phi$ and it is defined by

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

The gradient of a scalar function gives vector function

Q→1 If $\vec{r}$ be the radius vector, find ∇r^2.

⇒ solution,

we have,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

or, $r^2 = x^2 + y^2 + z^2$

Then,

$$\nabla r^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= \hat{i}\frac{\partial(x^2+y^2+z^2)}{\partial x} + \hat{j}\frac{\partial(x^2+y^2+z^2)}{\partial y} + \hat{k}\frac{\partial(x^2+y^2+z^2)}{\partial z}$$

$$= \hat{i}\cdot 2x + \hat{j}\cdot 2y + \hat{k}\,2z$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2\vec{r}$$

Q. 2. If r be the position vector show that $\nabla r^n = n r^{n-2} \vec{r}$

⇒ Solution,

We have

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

or $r = \sqrt{x^2+y^2+z^2}$

or $r^n = \left(\sqrt{x^2+y^2+z^2}\right)^n$

$\therefore r^n = (x^2+y^2+z^2)^{n/2}$

Now,

$$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \hat{i}\frac{\partial (x^2+y^2+z^2)^{n/2}}{\partial x} + \hat{j}\frac{\partial (x^2+y^2+z^2)^{n/2}}{\partial y} + \hat{k}\frac{\partial (x^2+y^2+z^2)^{n/2}}{\partial z}$$

$$= \hat{i}\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.2x + \hat{j}\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.2y + \hat{k}\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}.2z$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}\vec{r}$$

$\therefore \nabla r^n = n r^{n-2}\vec{r}$

Proved

* **Physical signification of Gradient :-**

In order to explain the physical signification of gradient. Let us take electric potential as the scalar function.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Taking its gradient

$$\nabla V = \nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\nabla r^{-1}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[(-1)r^{-1-2}\vec{r}\right] \quad \left(\because \nabla r^n = n r^{n-2}\vec{r}\right)$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

$$= -E$$

It shows that electric field intensity is negative of potential gradient. Hence, the physical signification of gradient is that by using it we can find the value of electric field if we know the value of electric potential.

P. T. O

* **Divergence of vector function** :- If $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ be a vector differential operator and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be a vector function then divergence of that vector function is denoted by div $\vec{F}$ or $\nabla\cdot\vec{F}$. It is defined as $\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot$

$$(F_x\hat{i} + F_y\hat{j} + F_z\hat{k})$$

$$\therefore \nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Divergence of vector function is scalar.

q→ Find the divergence of position vector (radial vector).

⇒ solution,

we have,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then,

$$\nabla\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = 3 \text{ Ans}$$

* Physical signification of divergence :-

If the vector function is taken as the velocity of liquid then its divergence gives the rate at which liquid is crossing the given cross-section. If the divergence is positive, the liquid is expanding. If the divergence is negative, the liquid is contracting. And if divergence is zero, the liquid is incompressible.

Note :- If divergence of a vector function is zero the vector function is called solenoidal.

* Curl :- If $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ be the vector differential operator and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ be a vector function, then curl of this function is denoted by curl $\vec{F}$ or $\nabla \times \vec{F}$ and it is defined by $\nabla \times \vec{F}$.

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Q→ Find the curl of position vector.

⇒ Solution,

we have,

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Now,

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y\right) - \hat{j}\left(\frac{\partial}{\partial x} z - \frac{\partial}{\partial z} x\right) + \hat{k}\left(\frac{\partial}{\partial x} y - \frac{\partial}{\partial y} x\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

Note: If curl of any vector function is zero the vector function is irrotational.

* Physical signification of curl:

In order to explain the physical significance of curl consider the linear velocity ($\vec{v}$) as vector function.

we have,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega x & \omega y & \omega z \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(\omega y \cdot z - \omega z \cdot y) + \hat{j}(\omega z \cdot x - \omega x \cdot z) + \hat{k}(\omega x y - \omega y \cdot x)$$

Taking the curl of $\vec{v}$, we get

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ \omega y \cdot z - \omega z \cdot y & \omega z x - \omega x \cdot z & \omega x \cdot y - \omega y \cdot x \end{vmatrix}$$

$$= \hat{i}\left[\omega x - (-\omega x)\right] + \hat{j}\left(\omega y - (-\omega y)\right) + \hat{k}\left(\omega z - (-\omega z)\right)$$

$= 2\omega x\hat{i} + 2\omega y\hat{j} + 2\omega z\hat{k}$

$= 2(\omega x\hat{i} + 2\omega y\hat{j} + 2\omega z\hat{k})$

$= 2\vec{\omega}$

It shows that curl of linear velocity is equal to two times of angular velocity. Hence, the significance of curl is that it tranforms the linear vector rotational vector.

* Integration of vector:

1. Line integral : Line integral of a function is its integration along a curve(or line). For vector function $\vec{F}$ the line integral is represented as $\int \vec{F}\cdot d\vec{l}$

2. Surface integral : Surface integral of a function is its integration over the surface. For vector function $\vec{F}$, the surface integral is represented as $\int_s \vec{F}\,\vec{ds}$ or $\int_s \vec{F}\,dxdy$

3. Volume integral : Volume integral of a function is its integration over the volume. For vector function $\vec{F}$ the volume integral is represented as $\int_v \vec{F}\,dv$.

\# Note: In terms of surface integral and line integral, the divergence and curl defined as

$$\nabla\cdot\vec{F} = \lim_{\Delta V\to 0} \frac{1}{\Delta V}\int_s \vec{F}\,\vec{ds}$$

$$\nabla\times\vec{F} = \lim_{\Delta s\to 0} \frac{1}{\Delta s}\int \vec{F}\,\vec{dl}$$

Q→1. Prove that $\vec{A} = 3y^2z^2\hat{i} + 3x^2z^2\hat{j} + 3x^2y^2\hat{k}$ is solenoidal vector.

⇒ Solution,

$\vec{A} = 3y^2z^2\hat{i} + 3x^2z^2\hat{j} + 3x^2y^2\hat{k}$

Then,

$$\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(3y^2z^2\hat{i} + 3x^2z^2\hat{j} + 3x^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}3y^2z^2 + \frac{\partial}{\partial y}3x^2z^2 + \frac{\partial}{\partial z}3x^2y^2$$

$$= 0+0+0$$

$$= 0$$

It shows that the given vector is a solenoidal vector.

Q→2. Find the constant so that the vector $\vec{A} = (x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}$ becomes solenoidal.

⇒ Solution,

$\vec{A} = (x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}$

Then,

$$\nabla\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x+3y)\hat{i} + (2y+3z)\hat{j} + (x+az)\hat{k}\right]$$

or, $0 = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(2y+3z) + \frac{\partial}{\partial z}(x+az)$

or, $0 = (1+0) + (2+0) + (0+a)$

or, $0 = 1+2+a$

or, $0 = 3+a$

$\therefore a = -3.$

Q→3. Show that the field $\vec{E} = 6xy\hat{i} + (3x^2 - 3y^2)\hat{j}$ is irrotational.

⇒ Solution,

Given, $\vec{E} = 6xy\hat{i} + (3x^2 - 3y^2)\hat{j}$

Then,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy & (3x^2-3y^2) & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(6x - 6x)$$

$$= 0 - 0 + 0$$

$$\therefore = 0$$

It shows that $\vec{E}$ is irrotational.

Q→4. Determine whether the electric field $\vec{E} = xy\hat{i} + y^3\hat{j}$ is conservative or not?

⇒ Solution,

Given, $\vec{E} = xy\hat{i} + y^3\hat{j}$

Then,

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & y^3 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Since, the given electric field is not conservative.

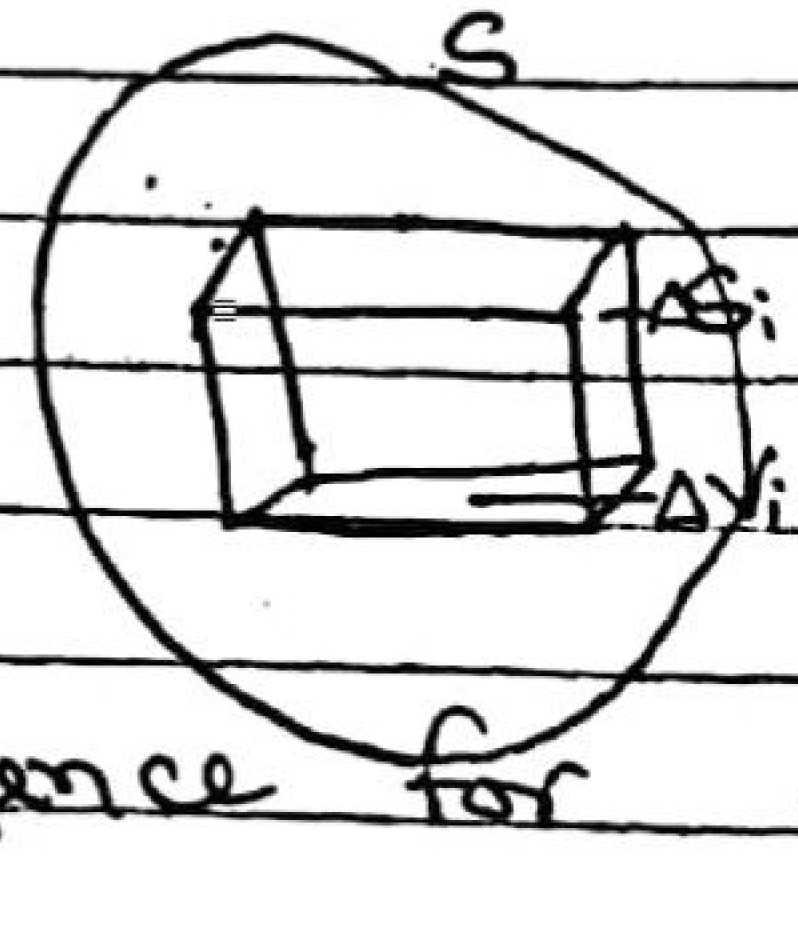
Gauss Divergence theorem:

Gauss divergence theorem relates the surface integral of a function with the volume integral. It states that, 'the surface integral of a function over a surface is equal to volume integral of divergence of that function over the volume inclosed by surface.

i.e. $\int_s \vec{F}\cdot\vec{ds} = \int_v (\nabla\cdot\vec{F})\,dv$

Proof:

Let $\vec{F}$ be a vector function defined over the surface 's' which encloses the volume V. Let us imagine an elementary region inside volume having volume ΔV_i and enclosed by surface ΔS_i as shown in fig.



Then, we can write

$$\int_s \vec{F}\cdot\vec{ds} = \sum \int_{\Delta S_i} \vec{F}\cdot\vec{ds} \longrightarrow (I)$$

Using the definition of divergence for surface ΔS_i, we get

$$\nabla\vec{F} = \lim_{\Delta V_i \to 0} \frac{1}{\Delta V_i} \int_{\Delta S_i} \vec{F}\,\vec{ds}.$$

or, $\int_{\Delta S_i} \vec{F}\vec{ds} = (\nabla\cdot\vec{F})\,\Delta V_i \quad (\text{for } \Delta V_i \to 0)$

Taking summation on both sides,

$$\sum \int_{\Delta S_i} \vec{F}\cdot\vec{ds} = \sum (\nabla\cdot\vec{F})\cdot \Delta V_i \quad (\text{for } \Delta V_i \to 0)$$

Since ΔV_i is very small the summation can be replaced by integration, so

$$\sum \int_{\Delta S_i} \vec{F}\cdot\vec{ds} = \int_V (\nabla\cdot\vec{F})\,dv \longrightarrow (II)$$

From eq^n (I) & (II)

$$\int_S \vec{F}\cdot\vec{ds} = \int_V (\nabla\cdot F)\,dv$$

It shows that the gauss divergence theorem.

(I)

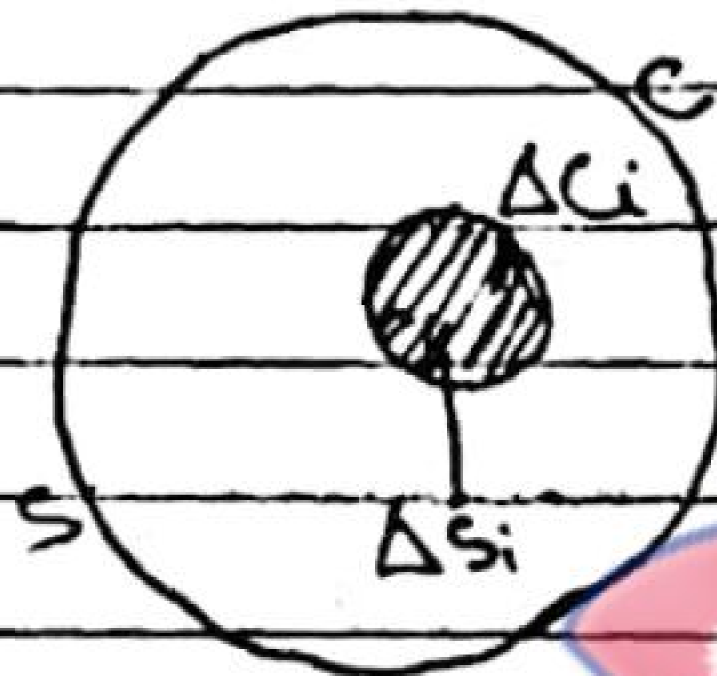
* Stoke theorem:

Stoke theorem relates the line integral of a vector function with surface integral. It states that, the line integral of a vector function over a closed path is equal to surface integral of curl of that vector function over the surface enclosed by that path.

i.e. $\oint \vec{F}\,\vec{dl} = \int_S (\nabla\times\vec{F})\cdot\vec{ds}$

Proof:

let $\vec{F}$ be a vector function defined along the curve c which contains the surface s. let us imagine an elementary surface ΔS_i enclosed by curve ΔC_i shown in fig.



Then we can write,

$$\oint \vec{F}\cdot\vec{dl} = \sum \int_{\Delta C_i} \vec{F}\vec{dl} \rightarrow (I)$$

From definition of curl for the elementary region we can write

$$\nabla \times \vec{F} = \lim_{\Delta S_i \to 0} \frac{1}{\Delta S_i} \int_{\Delta C_i} \vec{F}\vec{dl}$$

or, $$\int_{\Delta C_i} \vec{F}\vec{dl} = (\nabla \times \vec{F})\Delta S_i \ (\text{for } \Delta S_i \to 0)$$

Taking summation both sides

$$\sum \int_{\Delta C_i} \vec{F}\cdot\vec{dl} = \sum (\nabla \times \vec{F})\Delta S_i \ (\text{for } \Delta S_i \to 0)$$

Since, ΔS_i is very small we can replace summation by integration.

So,

$$\sum \int_{\Delta C_i} \vec{F}\vec{dl} = \int_s (\nabla \times \vec{F})\cdot\vec{ds}) \longrightarrow (II)$$

From eq^n (I) and (II)

$$\oint \vec{F}\cdot\vec{dl} = \int_s (\nabla \times \vec{F})\cdot\vec{ds}$$

It shows that stokes theorem

* **Green's theorem**:

If S be the closed region in xy plane bounded by a simple closed curve C and M and N are continuous function of x and having continuous derivatives then,

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$ where

the curve C, transversed in the anticlockwise direction. Green's theorem in a plane is a special case of stoke's theorem.

proof:

Let S be a closed region in xy plane bounded by a closed curve C. Suppose $\vec{A}$ is a vector field having M and N as its x and y components respectively. then

$$\vec{A} = M\hat{i} + N\hat{j} \longrightarrow (I)$$

In the xy plane a displacement vector $d\vec{r}$ is given by

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} \longrightarrow (II)$$

According to stoke's theorem

$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S (\nabla\times\vec{A})\cdot ds \longrightarrow (III)$$

From (I), (II) and (III)

$$\vec{A}\cdot d\vec{r} = (M\hat{i} + N\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$= M\,dx + N\,dy \longrightarrow (IV)$$

Also,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\frac{\partial N}{\partial z}\right) + \hat{j}\frac{\partial M}{\partial z} + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \longrightarrow (v)$$

Consider a small area element $d\vec{s}$ on the surface S. As S lies in the xy plane the area vector will point in the z-direction. As the curve C is tranversed in the anticlockwise direction

$$\therefore d\vec{s} = ds\,\hat{k} \longrightarrow (vi)$$

From (v) and (vi)

$$(\nabla \times \vec{A}) \cdot d\vec{s} = \left[-\hat{i}\frac{\partial N}{\partial z} + \hat{j}\frac{\partial M}{\partial z} + \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] ds\,\hat{k}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) ds \longrightarrow (vii)$$

Also, $ds = dx\,dy \longrightarrow (viii)$

Substituting the value of $\vec{A} \cdot d\vec{r}$ from eqn. (iv)

$(\nabla \times \vec{A}) \cdot d\vec{s}$ from eqn. (vii) and ds from eqn (viii) in stoke's theorem we get,

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$ which is known as Green's theorem in plane.