Chapter-3 Gravitational potential and field

Kepler's laws

Kepler's laws on the basis of observation on the planetary movement formulated three basic laws mentioned below:

1. 1st law (law of orbit) :- It states that the planets revolved round the sun along elliptical orbit with the sun at one of this foci.

proof:

Consider a point p and Q be the perihelion (point closest to the sun) and aphelion (point farthest from the sun) respectively, where planet of mass m has orbital velocity v_1 and v_2. Then from the law of conservation of angular momentum

$m r_1 v_1 = m r_2 v_2$, where r_1 and r_2 be the perp distances from the sun on line of action of orbital velocities at P and Q respectively.

or, $r_1 v_1 = r_2 v_2$

since, $r_1 < r_2 \Rightarrow v_1 > v_2$

Which shows that the planet has greatest orbital velocity at perihelion and smallest orbital ~~ot~~ velocity at aphelion.

2. 2nd law (law of areal velocity) :- It states that the radius vector joining the sun to planet sweeps equal area in equal intervals of time.

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Proof:

Through orbital velocity of a planet is not same every where along its elliptical but its areal velocity (rate by which area is swept) remains constant.

Consider a planet moves from point P to Q in infinitesimally small time $dt \to 0$ such that tangential distance $PQ = dr \to 0$, Now area swept by radius vector is given by

$$dA = \frac{1}{2} SP \times PQ$$

$$dA = \frac{1}{2} r\, dr$$

$$\text{Areal velocity} \left(\frac{dA}{dt}\right) = \frac{1}{2} \frac{r\, dr}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} rv$$

$\left[\because \frac{dr}{dt} = v \text{ instantaneous velocity}\right]$

Put $mvr = L$ (angular momentum)

$$rv = \frac{L}{m}$$

$$\therefore \frac{dA}{dt} = \frac{1}{2} \frac{L}{M} = \text{constant}$$

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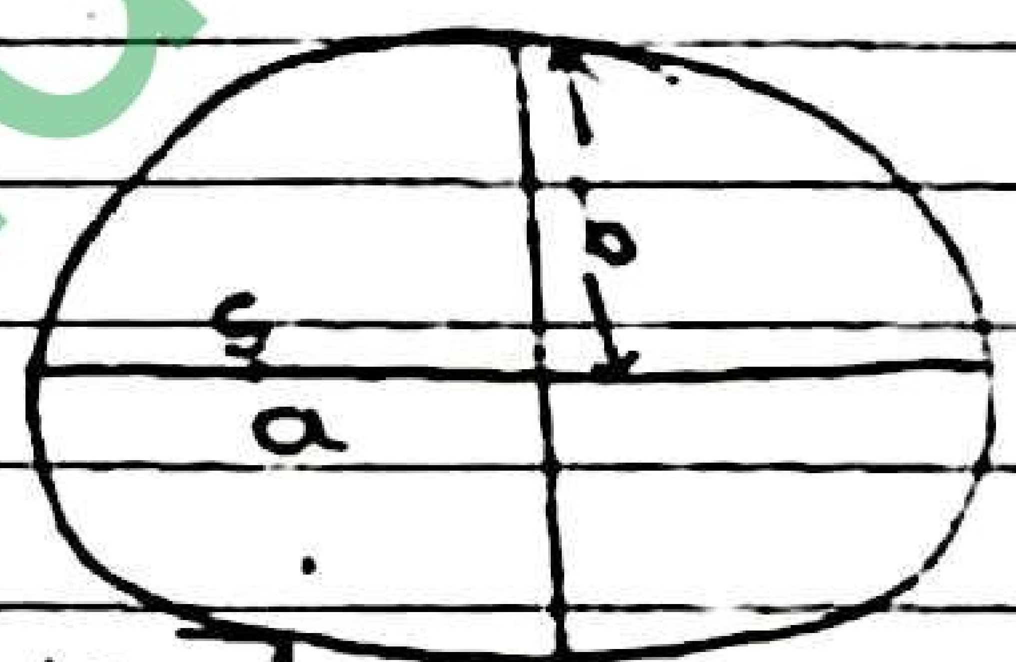
3. 3rd law (law of period) :- It states that square of time period of a satelite is directly proportional to cube of semimajor axis of the orbit of the planets.

Proof

Consider a and b are semi major and semininor axes of elliptical orbit of a planet under consideration then total area ~~ea~~ enclosed by orbit is given by

$A = \pi ab$



But time period

$$T = \frac{\text{Area}}{\text{Areal velocity}}$$

$$\text{or, } T = \frac{\pi ab}{L/2m} \quad \left[\because \frac{dA}{dt} = L/2m\right]$$

$$\text{or, } T = \frac{2\pi mab}{L}$$

$$\text{or, } T^2 = \frac{4\pi^2 m^2 a^2 b^2}{L^2}$$

If l be the semilatus rectum, then

$$l = \frac{b^2}{a} \Rightarrow b^2 = la$$

$$\therefore T^2 = \frac{4\pi^2 m^2 a^3 l}{L^2}$$

$$\text{or, } T^2 \propto a^3$$

Where, $\frac{4\pi^2 m^2 l}{L^2}$ is constant

Which proves the Kepler's 3rd law.

Gravitational force

The mutual force of attraction between any two material bodies of this universe due to their masses is called gravitation or gravitational force.

Newton's law of gravitation

It states that the mutual force of attraction between any two bodies of the universe is directly proportional to the product of their masses and inversely proportional ~~p~~ to the square of distance between their effective centres.

Mathematically,

$$F = \frac{G M_1 M_2}{r^2} \longrightarrow (1)$$

Where, M_1 and M_2 be the masses of the bodies under consideration ~~distuated~~ situated at central seperation r; G be a constant called universal gravitational constant. -ve sign simply indicates that the bodies attract each other towards their respective centres.

Value of universal gravitational constant

From eq^n^ (1)

$$G = \frac{F r^2}{M_1 M_2}$$

Taking $M_1 = M_2 = 1kg$ and $r = 1m$, we have

$G = F$ (numerically)
This shows that universal gravitational constant is numerically equal to force of attraction between two bodies of masses 1 Kg each kept at central seperation 1m. So the value of G is very small. SI unit of G is Nm^2Kg^{-2} and dimensional formula is $[M^{-1}L^3T^{-2}]$. The accept value of G is $6.67\times10^{-11}\ Nm^2Kg^{-2}$ and this value remains same everywhere independent on the type of interacting bodies and intervening medium.

Gravitational field
The space around a body within which its gravitational force of attraction can be experienced is called the gravitational field. The influence may vary from point to point and can measured as gravitational field intensity.

Gravitational field intensity
Gravitational field intensity at a point within the gravitational field is defined as the force experienced by a unit test mass kept at that point. It is denoted by E
If a test mass m experiences force F when kept at distance r from the influencing body of mass M then gravitational field

intensity E can be expressed as

$$E = \frac{F}{m} \longrightarrow (1)$$

SI unit of gravitational field intensity is NKg^{-1} and dimensional formula is $[LT^{-2}]$.

But $F = -\frac{GMm}{r^2}$

From (1)

$$E = -\frac{GMm}{mr^2}$$

$$\therefore E = \frac{GM}{r^2} \longrightarrow (2)$$

-ve sign indicates that gravitational field intensity is directed towards the centre of influencing body.

Gravitational potential

The workdone in moving a test body of unit mass from infinity to a point in the gravitational field of another influencing body is called gravitational potential at that point due to ~~influcing~~ influencing body. It is denoted by V.

If W be the workdone while bringing a test body of mass m from infinity to a point within the gravitational field of another influencing body then gravitational potential at the point under consideration can be mathematically defined as

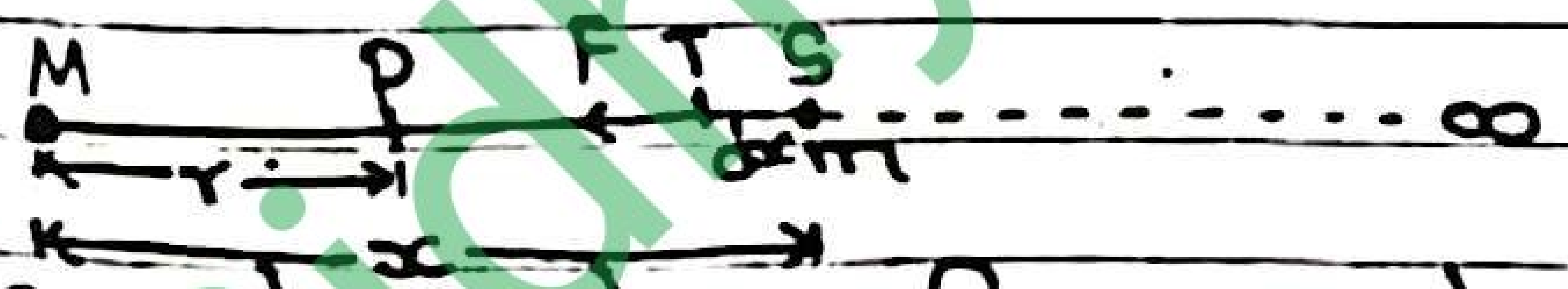
$$V = \frac{W}{m}$$

The SI unit of gravitational potential is Jkg^{-1} and dimensional formula is $[M^0L^2T^{-2}]$. It is a scalar quantity.

Gravitational potential at a point due to point mass

Consider a point mass M and a test body of mass m being carried from infinity to a certain point P lying at distance r from the point mass.

Let at any intermediate instant the test body arrives at point S at distance x from point mass as shown in figure.



So, at point 'S' force on test mass due to point mass, according to the Newton's law of gravitation is given as

$$F = -\frac{GMm}{x^2}$$

[-ve sign indicates that force is towards M]

If the test body further moves infinitesim" small distance dx from S to T then small workdone dW can be expressed as:

$$dW = -\frac{GMm}{x^2}(-dx)$$

$$\therefore dW = \frac{GMm}{x^2} dx \longrightarrow (1)$$

Now, total workdone on the test body can be found by integrating eq^n (1) within limits $x = \infty$ to r

i.e. $$W = \int_{\infty}^{r} \frac{GMm}{x^2} dx$$

or, $$W = GMm \int_{\infty}^{r} x^{-2} dx$$

or, $$W = GMm \left[\frac{x^{-1}}{-1}\right]_{\infty}^{r}$$

or, $$W = -GMm \left[\frac{1}{x}\right]_{\infty}^{r}$$

or, $$W = -\frac{GMm}{r} \longrightarrow (2)$$

This gives total workdone on test body called gravitational potential energy.

i.e. gravitational potential energy (U) $= -\frac{GMm}{r}$

so, Gravitational potential, $V = \frac{W}{m}$

or, $V = \frac{GM}{r}$ ⟶ (3)

Which is the required expression of gravitational potential.

Gravitational potential Gradient

The rate of change of gravitational potential within gravitational field with respect to distance is known as gravitational potential gradient.

If gravitational ~~potention~~ potential change by dV, for change in distance by dr, then,

Gravitational potential gradient $= \frac{dV}{dr}$

SI unit of potential gradient is $JKg^{-1}m^{-1}$ equivalently NKg^{-1}

Relation betn potential gradient & electric field intensity

Consider an influencing body of mass M and a point lying at distance r from the body where electric field intensity is given by

$E = -\frac{GM}{r^2}$ ⟶ (1)

Also, gravitational potential at that point becomes:

$$V = -\frac{GM}{r}$$

Differentiating w.r.t. to r, we get

$$\frac{dV}{dr} = -GM\frac{d}{dr}(r^{-1})$$

$$or, \frac{dV}{dr} = -GM\left(-\frac{1}{r^2}\right)$$

$$or, \frac{dV}{dr} = \frac{GM}{r^2}$$

$$or, -\frac{dV}{dr} = -\frac{GM}{r^2}$$

$$or, E = -\frac{dV}{dr} \longrightarrow (2)$$

Which is the relation between potential gradient and electric field intensity.

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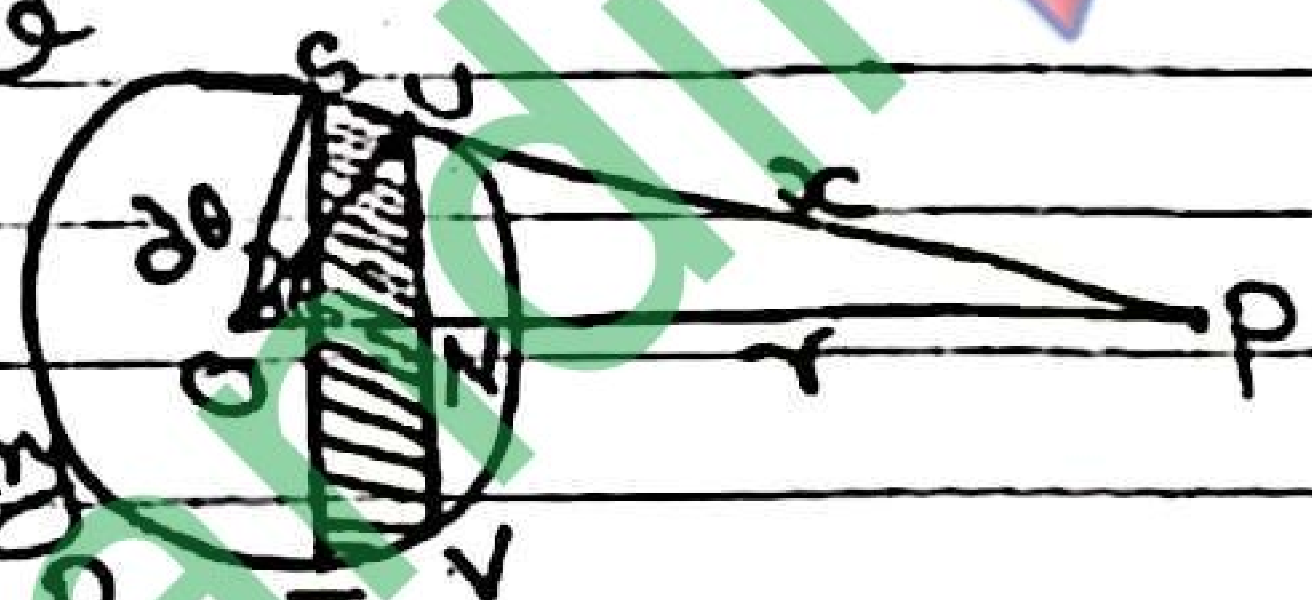
Gravitational potential & Gravitational field intensity due to thin spherical shell

Consider a thin spherical shell of mass M, radius R, centred O. Let σ be the surface density, such that

$$\sigma = \frac{M}{4\pi R^2}$$

Let's cut this hollow sphere by two close planes ST and UV, so that we get an elementary ring of radius UN. Let P be the point under consideration at distance r from the centre of the spherical shell such that $\angle UOP = \theta$ and $\angle SOU = d\theta \to 0$ and $UP = x$.



Here, radius of elementary ring $(UN) = OU \sin\theta$
$= R\sin\theta$

Thickness of elementary ring $(SU) = R\,d\theta$

Circumference of ring $= 2\pi R\sin\theta$

Surface area of ring $= 2\pi R\sin\theta \cdot R\,d\theta$
$= 2\pi R^2 \sin\theta\, d\theta$

Mass of ring $(dm) = \sigma \times 2\pi R^2 \sin\theta\, d\theta$

or, $dm = \frac{M}{4\pi R^2} \times 2\pi R^2 \sin\theta\, d\theta$

$$dm = \frac{1}{2} M \sin\theta\, d\theta \longrightarrow (1)$$

This gives the mass of elementary ring. Since all the points of elementary are at same distance x from point P under consideration, then the gravitational potential at point P due to elementary ring only is given by

$$dV = -\frac{G(dm)}{x}$$

or, $dV = -\frac{1}{2} GM \frac{\sin\theta d\theta}{x} \longrightarrow (2)$

From geometry,

$$UP^2 = UN^2 + NP^2$$

or, $x^2 = (R\sin\theta)^2 + (OP-ON)^2$

or, $x^2 = R^2\sin^2\theta + OP^2 + ON^2 - 2OP \times ON$

or, $x = R^2\sin^2\theta + r^2 + R^2\cos^2\theta - 2rR\cos\theta$

or, $x = r^2 + R^2 - 2rR\cos\theta$

Differentiating, we get

$2xdx = 0 + 0 - 2rR(-\sin\theta) d\theta$

or, $2xdx = 2rR\sin\theta d\theta$

or, $\sin\theta d\theta = \frac{xdx}{rR}$

So eqn. (2) becomes,

$$dV = -\frac{1}{2} \frac{GM}{rR} dx \longrightarrow (3)$$

Now gravitational potential due to entire spherical shell can be found by integrating eqn. (3)

Case-I :- If point p lies outside the spherical shell $(r > R)$

In this case eqn. (3) should be integrated within limits $x = r-R$ to $r+R$

$$\text{i.e. } V = \int_{r-R}^{r+R} -\frac{1}{2}\frac{GM}{rR}dx$$

$$\text{or, } V = -\frac{GM}{2Rr}\int_{r-R}^{r+R} dx$$

$$\text{or, } V = -\frac{GM}{2Rr} \times 2R$$

$$\therefore \boxed{V = -\frac{GM}{r}} \longrightarrow (4)$$

Also,
electric field

$$\boxed{E = -\frac{dV}{dr} = -\frac{d}{dr}\left[-\frac{GM}{r}\right]}$$

$$\text{or, } E = GM\frac{d}{dr}\left[\frac{1}{r}\right]$$

$$\text{or, } \boxed{E = -\frac{GM}{r^2}} \longrightarrow (5)$$

Case - II :- If point p lies on the surface of spherical shell $(r = R)$:

In this case eq^n (3) should be integrated within limits $x = 0$ to $2R$, i.e.

$$V = \int_0^{2R} -\frac{1}{2}\frac{GM}{rR}dx$$

or, $$V = -\frac{GM}{2rR}\int_0^{2R} dx$$

or, $$V = -\frac{GM}{2rR} \times 2R$$

$$\therefore V = -\frac{GM}{r} = -\frac{GM}{R} \longrightarrow (6)$$

$[\because r = R]$

Also,

$$E = -\frac{dV}{dr} = -\frac{GM}{r^2} = -\frac{GM}{R^2} \longrightarrow (7)$$

Case - III :- If point p lies inside the spherical shell $(r < R)$

In this case eq^n (3) should be integrated within limits $x = R-r$ to $R+r$

i.e. $$V = \int_{R-r}^{R+r} -\frac{1}{2}\frac{GM}{rR}dx$$

or, $V = -\frac{GM}{2Rr} \times 2r$

$\therefore V = -\frac{GM}{R}$

Also,

$E = -\frac{dV}{dr} = 0$

Graphical illustration

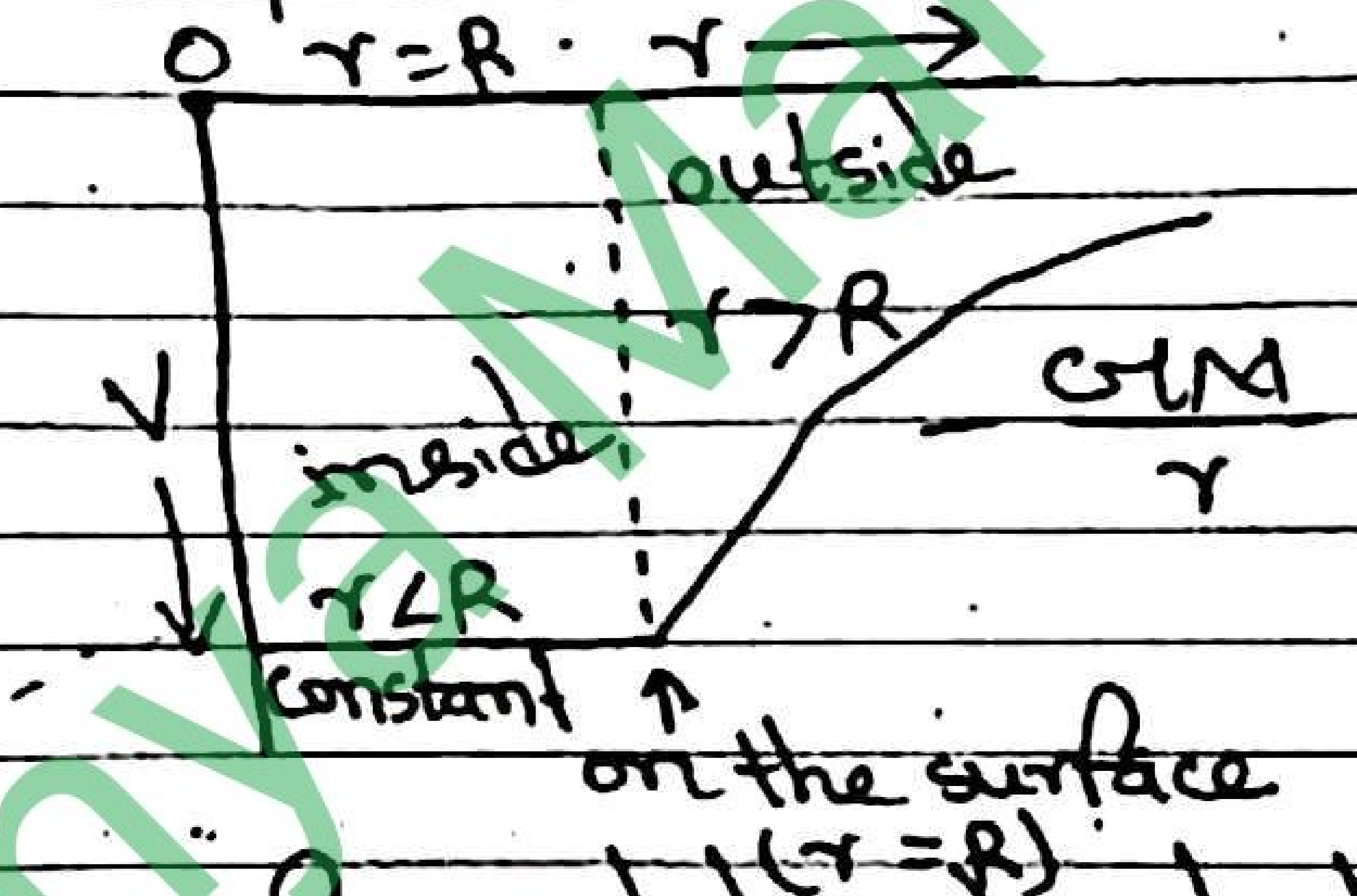
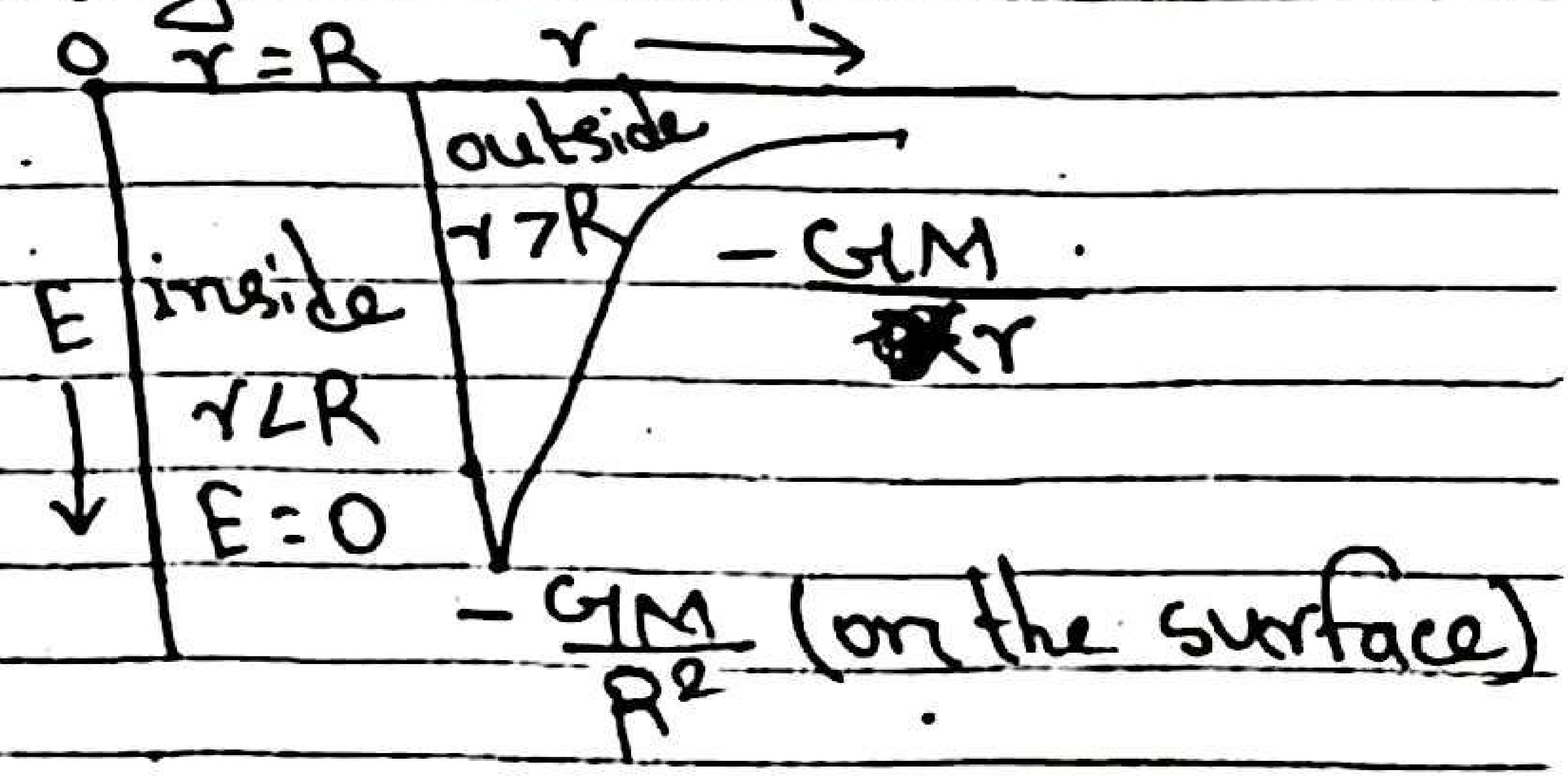


Fig. Variation of gravitational potential with 'r'



Gravitational potential and Gravitational field intensity due to a solid sphere.

(I) At a point outside the solid sphere:

Consider a solid sphere of mass M, radius R and density ρ centred at O so that

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

Consider any point P outside the sphere at distance r from the centre of the sphere, where gravitational potential & gravitational field are to be found.

The solid sphere can be supposed to be made up of large number of concentric and overlapping spherical shells having radii varying from zero to R.

Let's take one such spherical shell of radius x and thickness $dx \to 0$.

Here:

Surface area of shell $= 4\pi x^2$

Volume of shell $= 4\pi x^2 dx$

Mass of shell (dm) $= \rho \times 4\pi x^2 dx$

or, $dm = \frac{M}{4/3 \pi R^3} \times 4\pi x^2 dx$

$\therefore dm = \frac{3Mx^2 dx}{R^3}$

So, gravitational potential due to this elementary shell at point P is given by

$dV = -\frac{G(dm)}{r}$

or, $dV = \frac{-3GMx^2 dx}{rR^3} \longrightarrow (1)$

Now, gravitational potential due to entire solid sphere can be found by integrating eqn (1) within limits $x = 0$ to R.

i.e. $V = -\frac{3GM}{rR^3} \int_0^R x^2 dx$

or, $V = -\frac{3GM}{rR^3} \times \left[\frac{x^3}{3}\right]_0^R$

or, $V = -\frac{3GM}{rR^3} \times \frac{R^3}{3}$

$\therefore V = -\frac{GM}{r} \longrightarrow (2)$

Also,

$E = -\frac{dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) = -\frac{GM}{r^2} \longrightarrow (3)$

(II) If point p lies on the surface then $r = R$, so

$$V = -\frac{GM}{R} \quad \text{and} \quad E = -\frac{GM}{R^2} = -\frac{GM}{r^2}$$

(III) If point p lies inside the solid sphere

Consider point p lying inside the solid sphere at distance r from its centre such that $r < R$. So the point p lies on the surface of inner solid sphere of radius r and inside the ~~sp~~ spherical shells of radii varying from r to R as shown.

Now gravitational potential at p due to inner solid sphere of radius r is given by

$$V_1 = -\frac{G\left(\frac{4}{3}\pi r^3 \rho\right)}{r}$$

or $V_1 = -\frac{4}{3} G \pi r^2 \rho \longrightarrow (1)$

To find gravitational potential at p, due to other spherical shells having radii varying from r to R, let's take an elementary shell of radius x and thickness dx such that

mass of elementary shell (dm) $= 4\pi x^2 dx$

Gravitational potential due to this elementary shell at point p is

$$dV = -\frac{G\,dm}{x}$$

or, $$dV = -\frac{G\,4\pi x^2 dx\,\rho}{x}$$

$$\therefore dV = -4\pi G\rho x\,dx \longrightarrow (2)$$

So gravitational potential at p due to all outer shells of radii from r to R can be found by integrating eq^n (2) within limits $x = r$ to R

i.e. $$V_2 = \int_r^R -4\pi G\rho x\,dx$$

or, $$V_2 = -4\pi G\rho\left[\frac{x^2}{2}\right]_r^R$$

or, $$V = -4\pi G\rho \times \left(\frac{R^2 - r^2}{2}\right)$$

or, $$V = -2\pi G\rho\,(R^2 - r^2) \longrightarrow (3)$$

Now, total potential at p is given as

$$V = V_1 + V_2$$

$$= -\frac{4}{3}G\pi r^2\rho - 2\pi G\rho\,(R^2 - r^2)$$

$$= -\frac{2}{3}\pi\rho G\left(2r^2 + 3R^2 - 3r^2\right)$$

$$= -\frac{2}{3}\pi\rho G\left(3R^2 - r^2\right)$$

$$= -\frac{2}{3}\pi G\frac{M}{\frac{4}{3}\pi R^3}\left(3R^2 - r^2\right)$$

$$= -\frac{GM(3R^2 - r^2)}{2R^3}$$

$$\therefore V = -GM\left[\frac{3R^2 - r^2}{2R^3}\right] \longrightarrow (4)$$

This gives the total gravitational potential due to a solid sphere at a point inside it.

Now, electric field intensity is given by $E = -\frac{dV}{dr}$

$$= -\frac{d}{dr}\left(-GM\left[\frac{3R^2 - r^2}{2R^3}\right]\right)$$

$$= \frac{GM}{2R^3}(-2r)$$

$$\therefore E = -\frac{GM}{R^3}r \longrightarrow (5)$$

$$\therefore E \propto r$$

** Special cases: If point P lies at the centre of solid sphere then $r = 0$, so that

$$V = -GM\left[\frac{3R^2}{2R^3}\right]$$

$$V = -\frac{3}{2}\frac{GM}{R}$$

And $E = 0$

Graphical illustration

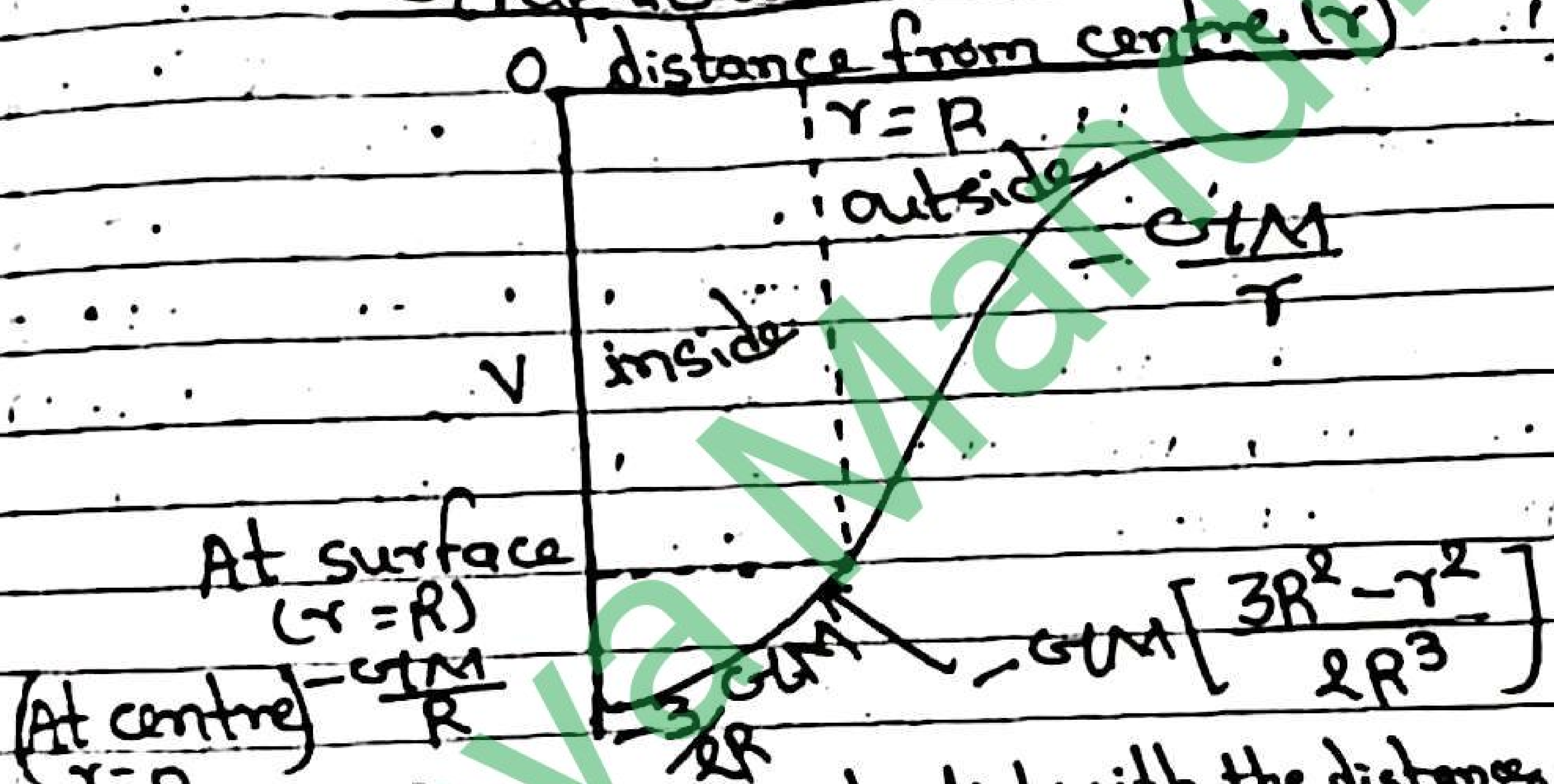


Fig. variation of gravitational potential with the distance from the centre of solid sphere

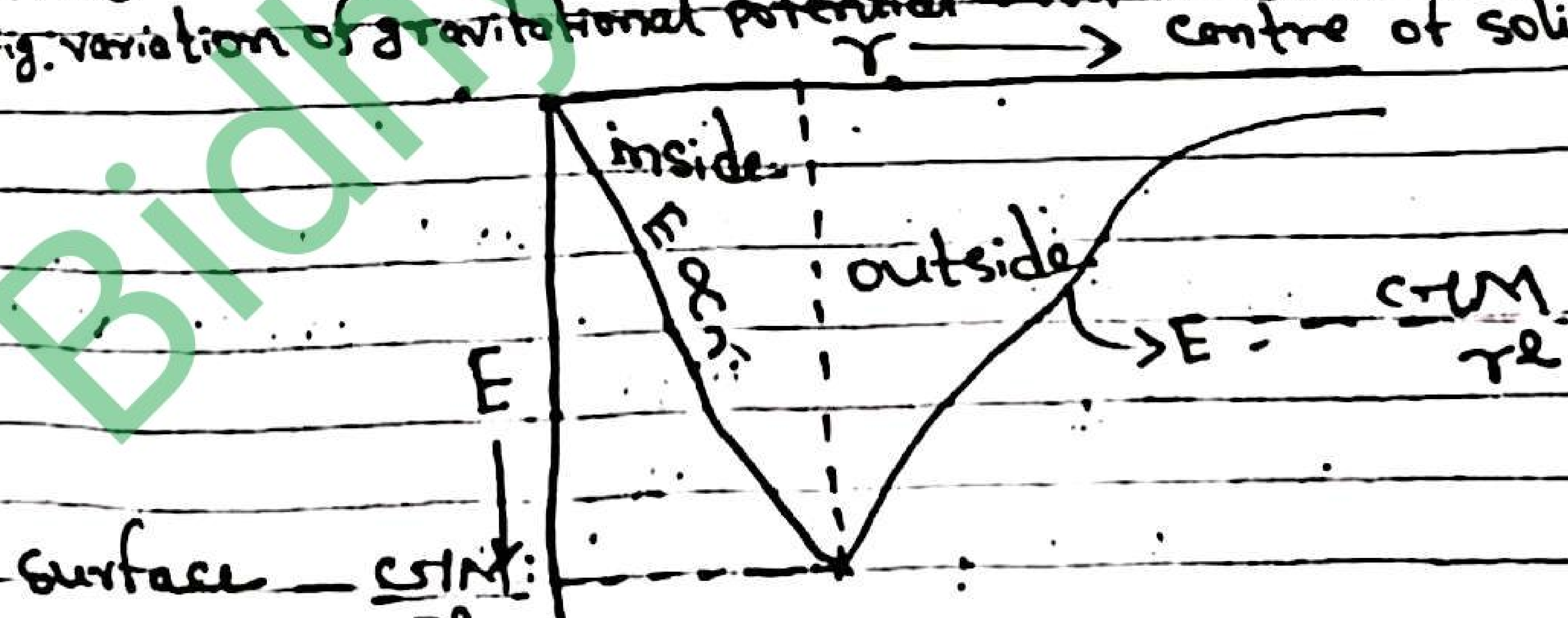


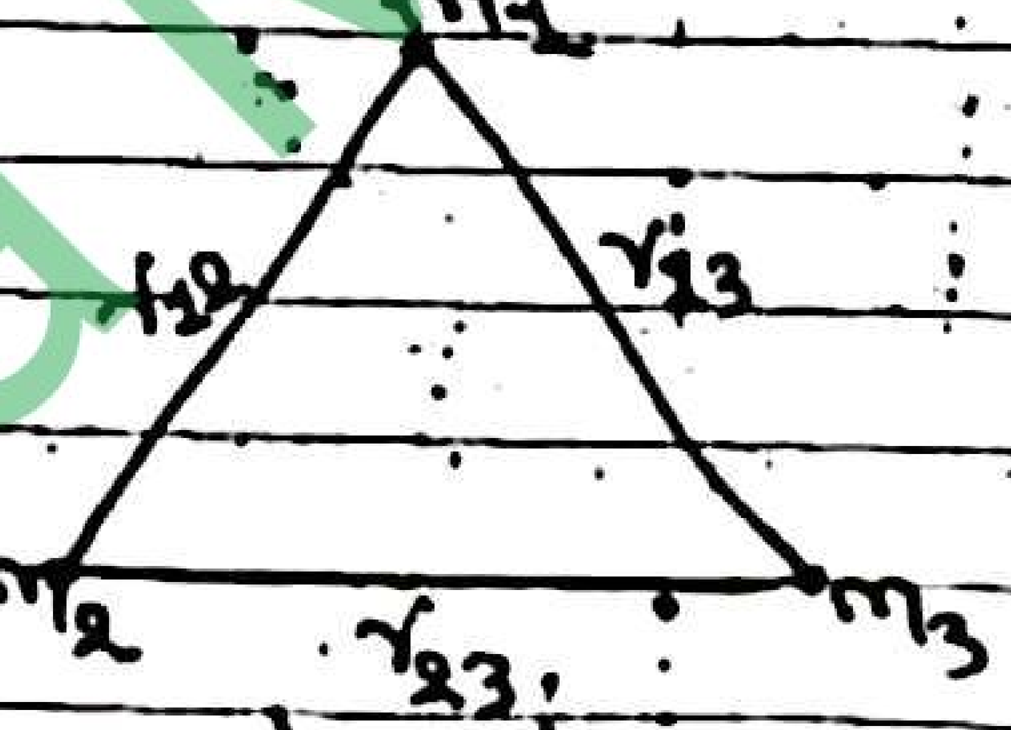
Fig: variation of gravitational field within the distance r from the centre of solid sphere.

Gravitational self energy

A large body can be supposed to be made up of large number elementary particles which intially were at infinite separation from each other.

Gravitational self energy of a body can be defined as the total workdone while bringing all its constituent particles intially from infinity to present configuration. It is usually denoted by U_s.

If m_1, m_2 and m_3 be the three particles system having present separation between m_1 and m_2 as r_{12}, m_1 and m_3 as r_{13} and m_2 and m_3 as r_{23} as shown in figure.



Here, workdone to being m_1 and m_2 at separation r_{12} from infinity is given as $W_{12} = -\frac{Gm_1 m_2}{r_{12}}$

Similarly, $W_{13} = -\frac{Gm_1 m_3}{r_{13}}$

$W_{23} = -\frac{Gm_2 m_3}{r_{23}}$

So total workdone or self energy becomes

$U_s = W_{12} + W_{13} + W_{23}$

or, $U_s = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$

or, $U_s = -\frac{1}{2}\left[\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_2}{r_{21}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_1m_3}{r_{31}} + \frac{Gm_2m_3}{r_{23}} + \frac{Gm_2m_3}{r_{32}}\right]$

In general for n particles system,

$U_s = -\frac{1}{2}G\sum_{i=1}^{n}\frac{m_im_j}{r_{ij}}$ for $i \neq j$

Gravitational self energy of a uniform sphere

Consider a body finally achieves spherical sphere of radius R, mass M and uniform density ρ such that

$\rho = \frac{M}{4/3\pi R^3} = \frac{3M}{4\pi R^3}$

Let at any indetermediate instant during formation of the body a sphere of radius x forms such that its mass becomes $\frac{4}{3}\pi x^3\rho$

Now, additional layer of thickness dx also forms around this sphere so that the mass of this layer becomes,

$4\pi x^2 dx\rho$

Now, gravitational self energy when a spherical layer of thickness dx forms on the sphere of radius x is given by

$$dU_s = -\frac{G\left(\frac{4}{3}\pi x^3\rho\right)\left(\frac{4}{3}\pi x^2 dx\rho\right)}{x}$$

$$\text{Or, } dU_s = -\frac{16}{3}G\pi^2\rho^2 x^4 dx \longrightarrow (1)$$

Now, total gravitational self energy associate with formation of solid sphere of radius R can be found by integrating eqn. (1) within limits $x=0$ to R.

$$\text{I.e, } U_s = -\frac{16}{3}G\pi^2\rho^2\int_0^R x^4 dx$$

$$\text{Or, } U_s = -\frac{16}{3}G\pi^2\rho^2\left[\frac{x^5}{5}\right]_0^R$$

$$\text{Or, } U_s = -\frac{16}{3}G\pi^2\rho^2\frac{R^5}{5}$$

$$\text{Or, } U_s = -\frac{16}{3}G\pi^2\left(\frac{3M}{4\pi R^3}\right)^2\times\frac{R^5}{5}$$

$$\text{Or } U_s = -\frac{16}{3}G\pi^2\times\frac{9M^2}{16\pi^2R^6}\times\frac{R^5}{5}$$

$$\therefore U_s = -\frac{3}{5}\frac{GM^2}{R} \longrightarrow (2)$$

This gives gravitational self energy associated with the formation of uniform sphere of mass M and radius R.

Gravitational self energy of Earth

We have, For earth

$M = 6 \times 10^{24}$ kg

$R = 6.4 \times 10^6$ m

$G = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2}

so, $U_s = -\frac{3}{5}\frac{GM^2}{R}$

Or, $U_s = -\frac{3}{5} \times 6.67 \times 10^{-11} \frac{(6 \times 10^{24})^2}{(6.4 \times 10^6)}$

$\therefore U_s = -2.25 \times 10^{32}$ J

The -ve sign indicates that amount of energy has been released during the formation of the earth.

Gravitational flux

Total gravitational field lines passing perp. through given area under consideration is called gravitational flux. It is denoted by ϕ.

If gravitational field E passes through surface area S making an angle θ with the normal on area then gravitational flux ϕ.

Can be mathematically expressed as

$\phi = E\cos\theta \cdot S$

or, $\phi = ES\cos\theta$

Or, $\phi = \vec{E}\cdot\vec{S}$

So, gravitational flux is the scalar product of gravitational field and ~~aso~~ area vector through which gravitational field passes.

* Special cases :

→ Case-1 : If gravitational field passes perpendicurly through given area 'A'.

i.e. $\theta = 0°$ then,

$\phi = EA\cos 0 = EA = \phi_{max}$

where A be the area.

→ Case-II : If gravitational field passes being tangential on given area

i.e. $\theta = 90°$ then

$\phi = EA\cos 90 = 0.$

Gauss law in gravitation

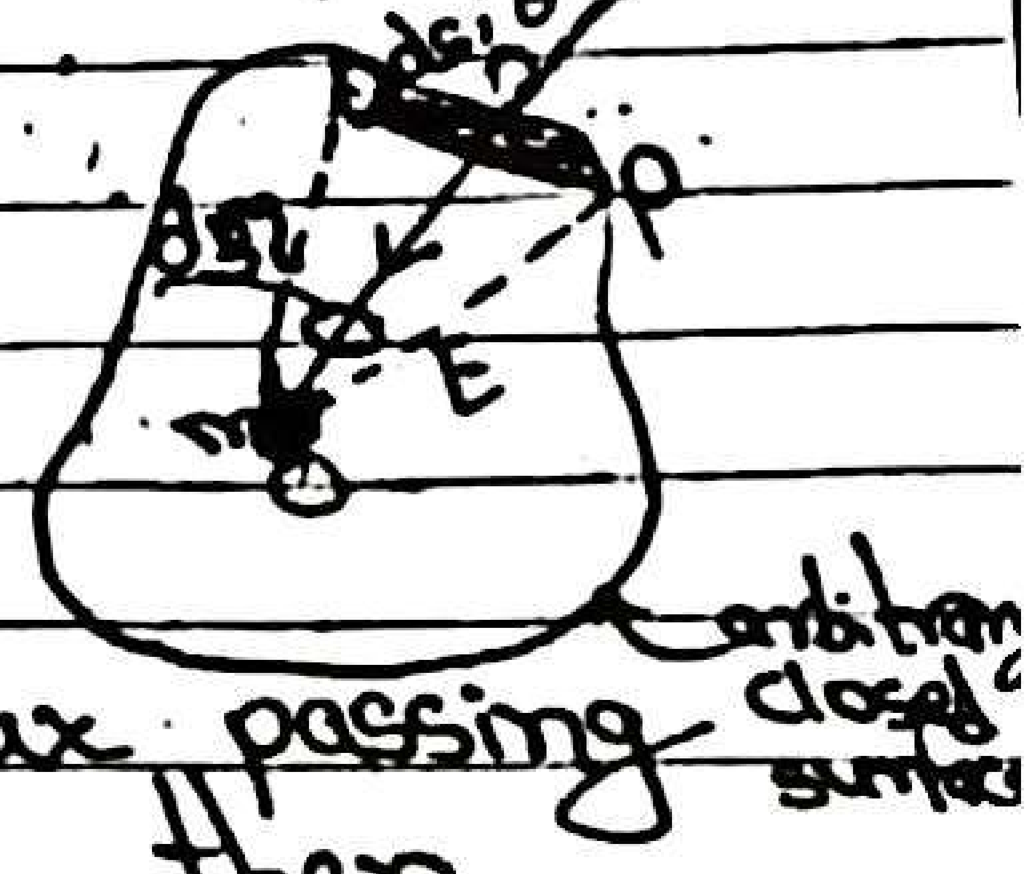
It states that the total gravitational flux through a closed surface is equal to $-4\pi G$ times mass enclosed within the surface. In short,

$$\phi = -4\pi G \times (\text{mass enclosed by the surface})$$

Proof

Suppose 'm' be the mass of a body at point 'O' enclosed within an arbitrary surface as shown in figure.

Let's take infinitesimally small area 'ds' at point 'p' at distance 'r' from 'O' where gravitational field intensity be E that makes angle θ with the normal on the elementary area.



Now,

$$E = -\frac{Gm}{r^2} \longrightarrow (1)$$

If $d\phi$ be the gravitational flux passing through the elementary area ds, then

Now,

$$d\phi = E ds \cos\theta$$

$$\text{or, } d\phi = -\frac{Gm}{r^2} ds\cos\theta$$

$$\text{Or, } d\phi = -Gm\left(\frac{ds\cos\theta}{r^2}\right)$$

$$\text{or, } d\phi = -Gm\, d\Omega \longrightarrow (2).$$

Where, $d\Omega = \frac{ds \cos\theta}{r^2}$ is solid angle subtended by elementary area at the point 'O'.
Now, total gravitational flux passing through entire closed surface can be obtained by integrating eqⁿ (2), i.e.

$$\phi = \int -Gm\, d\Omega$$

Or, $\phi = -Gm \int d\Omega$

Or, $\phi = -Gm\, 4\pi$

$\phi = 4\pi Gm \longrightarrow (3)$

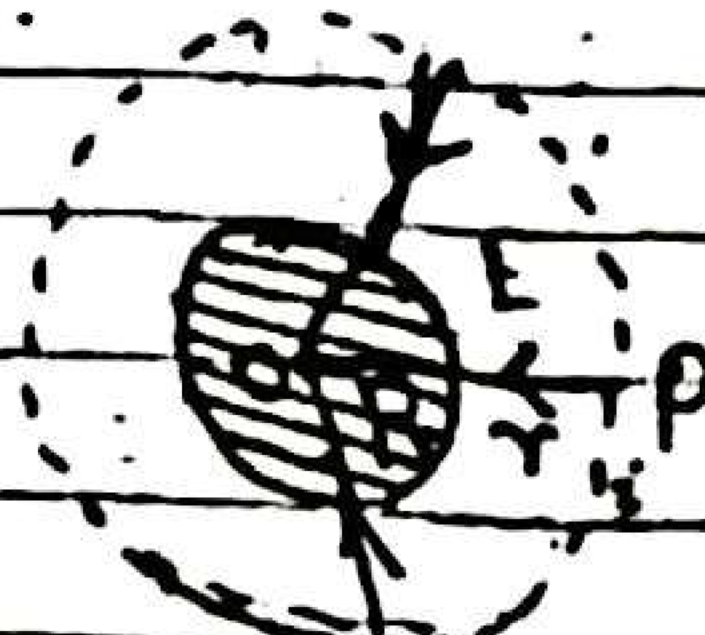
This proves Gauss law in gravitation.

Application of Gauss law

⟨I⟩ Gravitational field at a point due to a solid sphere

(a) At a point outside the solid sphere

Consider a solid sphere of mass M radius R centred at O with uniform density ρ. Let's take any point P outside the solid sphere at distance r from centre such that r > R where gravitational field has to be found as shown in figure.



To apply Gauss law, let's draw a spherical closed surface of radius r centred

'O' such that point p lies on the closed surface and gravitational field passes being perpendicular to the closed surface at every point. If ϕ be the gravitational flux passing through the closed surface, then

$$\phi = -4\pi GM \longrightarrow (1)$$

But, $\phi = E \times \text{surface area}(S)$

Or, $\phi = E \times 4\pi r^2 \longrightarrow (2)$

From eqn. (1) and (2),

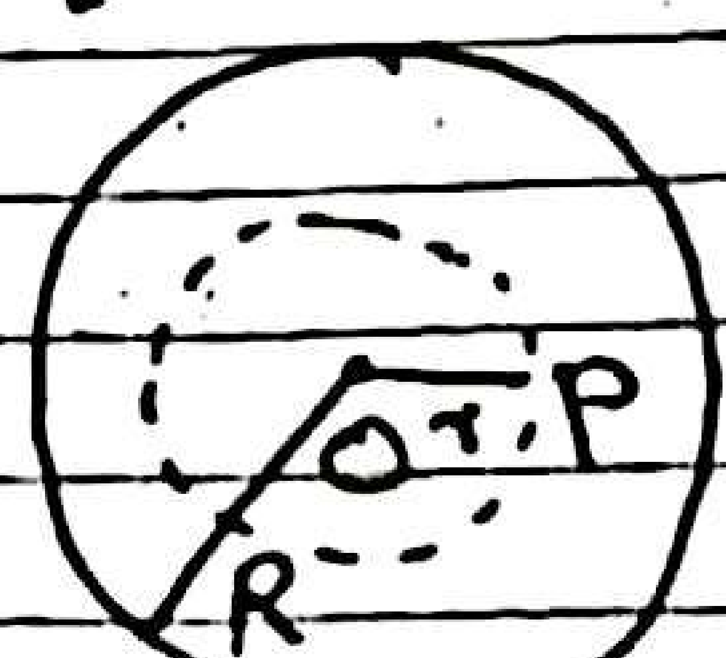
$$-4\pi GM = E \times 4\pi r^2$$

$$\text{or, } E = -\frac{4\pi GM}{4\pi r^2}$$

$$\therefore E = -\frac{GM}{r^2} \longrightarrow (3)$$

(b) At any point inside the solid sphere

Let's take any point p inside the solid sphere at distance r from the centre of the solid sphere such that R>r or r<R as shown in figure.



To apply gauss law, let's draw a closed spherical surface of radius 'r' centred at 'O' such that point p lies on the closed surface. If ϕ be the gravitational flux passing through the closed surface. If ϕ be the gravitational flux passing through the closed surface, then

$\phi = -4\pi G \times$ (mass enclosed)

Or, $\phi = -4\pi G \times \frac{4}{3}\pi r^3 \rho \longrightarrow (1)$

But from definition of gravitational flux

$\phi = E(4\pi r^2) \longrightarrow (2)$

Equating eq^n (1) and (2)

$$E(4\pi r^2) = -4\pi G \times \frac{4}{3}\pi r^3 \rho$$

$$\text{Or, } E = -\frac{4\pi G}{4\pi r^2} \times \frac{4}{3}\pi r^3 \rho$$

$$\text{Or, } E = -\frac{4\pi G r}{3} \times \left(\frac{3M}{4\pi R^3}\right)$$

$$\therefore E = \frac{GM}{R^3} r \longrightarrow (3)$$

This gives gravitational field inside the solid sphere.

* special case:- At centre of the solid sphere i.e. $r = 0$ then

$E = 0$.

(C) Gravitational field at any point on the surface of solid sphere, i.e. $r = R$, so,

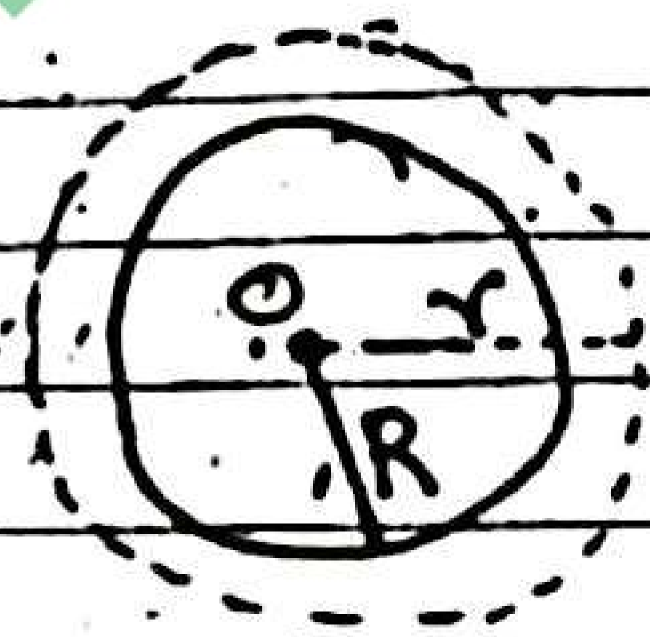
$$E = \frac{GM}{R^2}$$

(II) Gravitational field at a point due to a hollow sphere

(a) At any point outside the spherical shell

Consider a spherical shell of mass M, radius R and centred 'O'. Let P be any point outside the hollow sphere lying at distance r from the centre of hollow sphere where gravitational field has to be found using Gauss law. The situation is shown in figure.

To apply Gauss law, let's draw a spherical ~~shell~~ Gaussian surface of radius r concentric with the given hollow sphere such that point P underation lies on the Gaussian surface and gravitational field passes perp at every point on Gaussian surface.



If ϕ be the gravitational flux passing through the closed Gaussian surface then from Gauss law

$$\phi = -4\pi GM \longrightarrow (1)$$

But from definition of gravitational flux

$$\phi = E \times 4\pi r^2 \longrightarrow (2)$$

From eq^n (1) and (2), we get,

$$E \times 4\pi r^2 = -4\pi GM$$

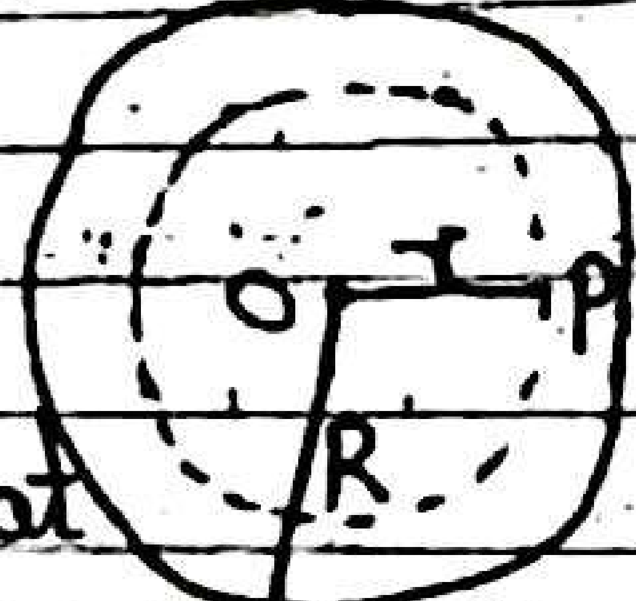
Or, $E = \frac{GM}{r^2}$ ——→ (3)

This gives gravitational field at a point outside the spherical shape body.

(b) <u>At any point inside the hollow sphere</u>

Consider point P to lie inside the hollow sphere at distance r from the centre of hollow sphere such that $r<R$. The situation is shown in fig.

To apply Gauss law, let's draw a spherical Gaussian surface of radius 'r' concentric with the hollow sphere such that point P lies on the Gaussian surface.



If ϕ be the gravitational flux through the Gaussian surface then from Gauss law,

$\phi = -4\pi G \times$ (mass enclosed

or, $\phi = 0$ [∵ no. of mass is enclosed]

But, $\phi = E \times 4\pi r^2$

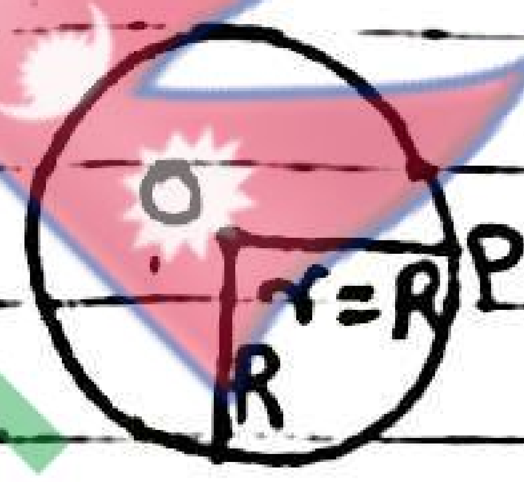
So, $E \times 4\pi r^2 = 0$

or, $E = 0$

This shows that gravitational field inside the hollow sphere is zero.

(c) At any point on the surface of hollow sphere

If point P lies on the surface of hollow sphere then the surface of the sphere itself acts at Gaussian surface and hence; $r = R$



So, $E = -\frac{GM}{r^2} = -\frac{GM}{R^2}$

For numerically,

Escape velocity, $V_e = \sqrt{2GM/R}$

Orbital velocity, $V_o = \sqrt{\frac{GM}{R}}$

Poisson's Equation

If a mass 'm' exist in a volume τ and ρ is density at a point inside the mass m then,

$$m = \int_V \rho d\tau \longrightarrow (1) \quad [\because dm = \rho d\tau]$$

If the volume τ is enclosed in a surface of area S, then from Gauss theorem,

$$\int_S \vec{E} \cdot \vec{ds} = \int_V \nabla \cdot E d\tau \longrightarrow (2)$$

Where $\vec{E}$ is gravitational field.
From Gauss law in gravitation

$$\oint_S \vec{E}\cdot\vec{ds} = -4\pi G m$$

Or, $$\oint_S \vec{E}\cdot ds = -4\pi G \int_V \rho\, d\tau \longrightarrow (3)$$
From eq^n (2) and (3), we get

$$\int_V \vec{\nabla}\vec{E}\, d\tau = -4\pi G \int_V \rho\, d\tau$$

Since, volume is taken arbitrarily,
$$\vec{\nabla}\cdot\vec{E} = -4\pi G \rho \longrightarrow (4)$$
But, $\vec{E} = -\nabla V$; where V is gravitational potential

Now, eq^n (4) becomes,
$$\nabla\cdot(-\nabla V) = -4\pi G \rho$$
$$\nabla^2 V = 4\pi G \rho \longrightarrow (5)$$
This is poisson's eq^n in gravitation.
Where, $\nabla^2 = \nabla\cdot\nabla$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$
Or, $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is called laplacian operator.