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Chapter - 4 Harmonic oscillator

A body or system that executes simple harmonic motion is called harmonic oscillator.

Simple Harmonic motion (SHM)

The periodic motion in which restoring force or accleration is directly proportional to displacement from mean position and always directed towards mean position is known as simple harmonic motion.

If F be the restoring force corresponding to displacement x then from definition of SHM, we have

$F \propto x$

Or, $F = -Kx \longrightarrow (1)$

Where, K be the constant and -ve sign indicates that the force always acts towards mean position. From Newton's 2nd law of motion $F = m\frac{d^2x}{dt^2} \longrightarrow (2)$

Equating (1) and (2), we get

$$m\frac{d^2x}{dt^2} = -Kx$$

$$\text{or, } \frac{d^2x}{dt^2} = -\left(\frac{K}{m}\right)x$$

$$\text{or, } \frac{d^2x}{dt^2} = -\omega^2 x \longrightarrow (3)$$

This gives accleration eqn. of SHM where $\omega = \sqrt{\frac{K}{m}}$ is angular frequency.

Multiplying both sides of eqn. (3) by $2\frac{dx}{dt}$

we get $2\frac{dx}{dt} \times \frac{d^2x}{dt^2} = -2\omega^2 x \frac{dx}{dt}$

Integrating both sides, we get

$$\int 2\frac{dx}{dt}\frac{d^2x}{dt^2} = \int -2\omega^2 x \frac{dx}{dt}$$

Or, $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + c$

Where c be the integrating constant

we know, for $x = a$ (amplitude)

$\frac{dx}{dt} = 0$

$\therefore 0 = -\omega^2 a^2 + c$

$\therefore c = \omega^2 a^2$

So, $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 a^2$

or, $\left(\frac{dx}{dt}\right)^2 = \omega^2(a^2 - x^2)$

$\therefore \frac{dx}{dt} = \omega\sqrt{(a^2 - x^2)} \longrightarrow (4)$

This gives velocity of SHM
further,

$$\frac{dx}{\sqrt{a^2-x^2}} = \omega dt$$

Integrating both side.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \omega dt$$

or, $\sin^{-1}\left(\frac{x}{a}\right) = (\omega t + \phi)$

Here ϕ be the integrating constant called phase constant of SHM.

$$\frac{x}{a} = \sin(\omega t + \phi)$$

$$x = a\sin(\omega t \pm \phi) \longrightarrow (5).$$

This represents eq^n of SHM.

Time period (T) of simple harmonic motion is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{K}{m}}} = 2\pi\sqrt{\frac{m}{K}}$

Also, the frequency (no. of vibration per sec) of oscillator is given as

$$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{\frac{m}{K}}} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}.$$

SHM in terms of angular accleration

Angular accleration of a body or system excuting rotatory SHM is directly proportional to angular displacement and it is directed towards mean position. So,
angular accleration (α) $\propto$ angular displacement (θ)

$$\alpha = -\omega^2\theta$$

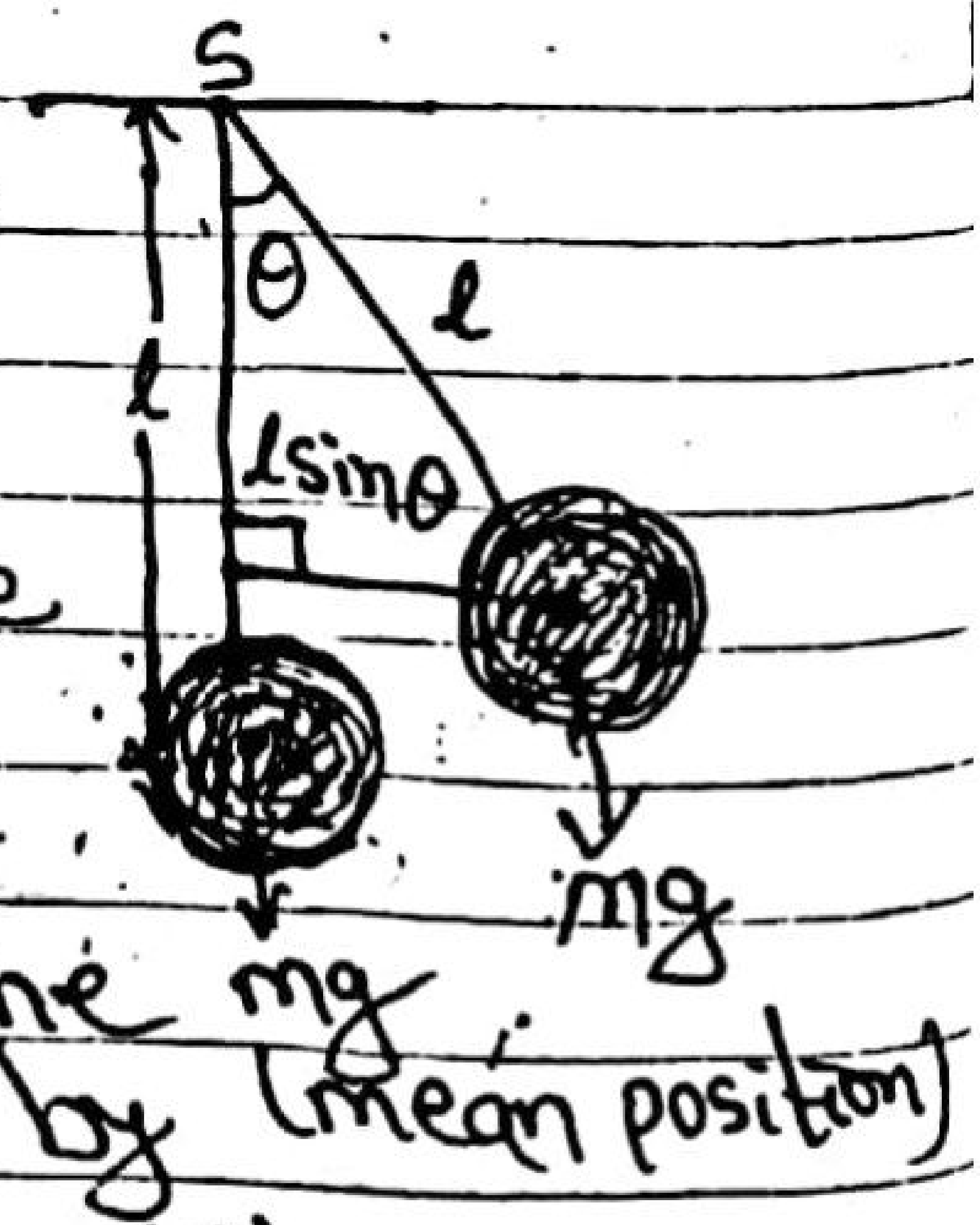
Second order differential form

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

Simple pendulum

A heavy point mass attached with perfectly flexible, inextensible and massless string capable to excute SHM is called simple pendulum.

Consider a bob of mass m suspended vertically having effective length l. When the bob is displaced slightly so that the string makes angle θ with vertical making perpendicular distance $l\sin\theta$ then restoring torque on the bob is mathematically given by



$$\tau = -mgl\sin\theta \longrightarrow (1)$$

(about an axis through point of suspension)
But,
$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

For small θ, higher power term can be neglected so, $\sin\theta \approx \theta$ then
eq^n (1) becomes

$\tau = -mgl\theta \longrightarrow (2)$

If this torque produced angular accleration $\alpha = \frac{d^2\theta}{dt^2}$ and I be the moment of inertia of the bob about axis through 'S' then,

$\tau = I\frac{d^2\theta}{dt^2} \longrightarrow (3)$

Now, equating (2) and (3), we get,

$I\frac{d^2\theta}{dt^2} = -mgl\theta$

or, $ml^2\frac{d^2\theta}{dt^2} = -mgl\theta$

$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \longrightarrow (4)$

Comparing this eq^n with general form

$\frac{d^2\theta}{dt^2} = -\omega^2\theta \longrightarrow (5)$

$$\omega = \sqrt{\frac{g}{l}}$$

Now, Time period (T) $= \frac{2\pi}{\omega}$

$$= 2\pi\sqrt{\frac{l}{g}}$$

Frequency of oscillation (f) $= \frac{1}{T}$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

Drawbacks of simple pendulum

1. Heavy point mass can not be realized in practice.
2. perfect flexible, inextensible and massless string can not be realized in real situation.
3. The motion will not be strictly simple harmonic angular displacement of simple pendulum becomes relatively larger.
4. The time period is valid only for small oscillation.

P.T.O.

Compound pendulum (physical pendulum)

A rigid body of any shape and size capable of oscillating about a vertical plane through horizontal axis is called compound pendulum. The oscillation of such pendulum for small angle displacement is SHM.

Consider a rigid body of mass m capable of oscillating about a vertical plane through horizontal axis passing from certain point s called point of suspension so that it lies at distance l from its centre of gravity G as shown in figure.

When the body is slightly displaced by small angle θ then centre of gravity gets shifted at G'. So that $SG' = l$ and perpendicular distance $l\sin\theta$. Now, for this displacement restoring torque on the body horizontal axis through s is given by

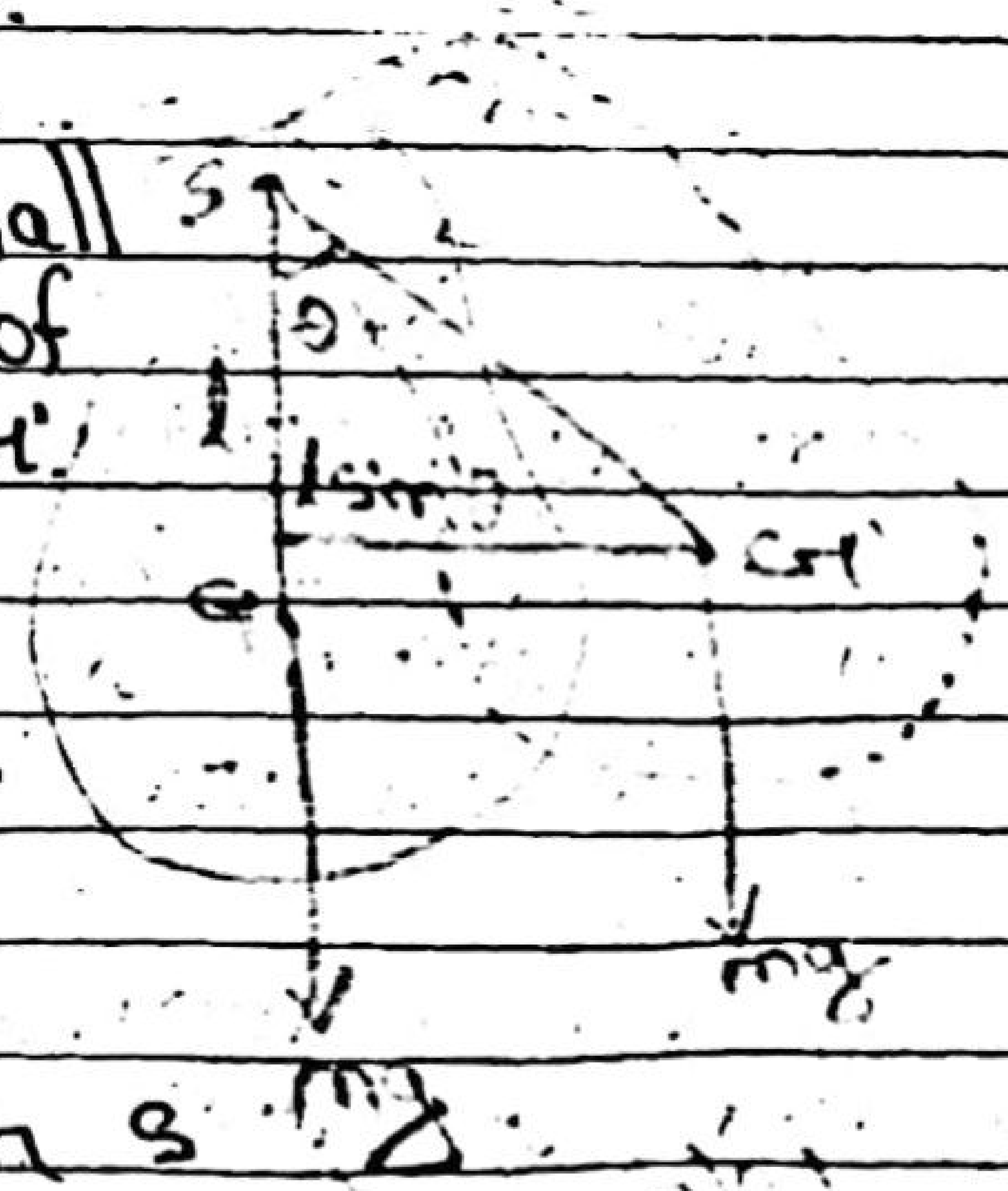


Fig: compound pendulum

$\tau = -mg\,l\sin\theta$

We know,

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$

For small θ, higher power term can be neglected. $\sin\theta \approx \theta$

$$\tau = -mgl\theta \longrightarrow (1)$$

If this torque produced angular accleration α on the body having moment of inertia about given axis (passing through s) I. Then

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \longrightarrow (2)$$

Equating (1) & (2), we get

$$I\frac{d^2\theta}{dt^2} = -mgl\theta$$

$$\text{or, } \frac{d^2\theta}{dt^2} = -\left(\frac{mgl}{I}\right)\theta \longrightarrow (3)$$

This shows that angular accleration is directly proportional to angular displacement θ and -ve sign indicates that angular accleration so as to decrease increasing so, the motion of compound pendulum is SHM.

Now comparing eq^n (3) with the general form of SHM

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

$$\omega = \sqrt{\frac{mgl}{I}}$$

Time period (T) $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgl}} \longrightarrow (4)$

Further,

If I_G be the moment of inertia of given compound pendulum about axis through centre of mass G then its moment of inertia about axis through S, 'I' ~~be~~ according to parallel axis theorem is given as

$$I = I_G + ml^2$$

Or, $I = MK^2 + ml^2$

Where, K be the radius of gyration of given compound pendulum about axis through G.

Now, eq^n (4) becomes

$$T = 2\pi\sqrt{\frac{m(K^2+l^2)}{mgl}}$$

$$\text{Or, } T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}}$$

$$\text{Or, } T = 2\pi\sqrt{\frac{L}{g}} \longrightarrow (5)$$

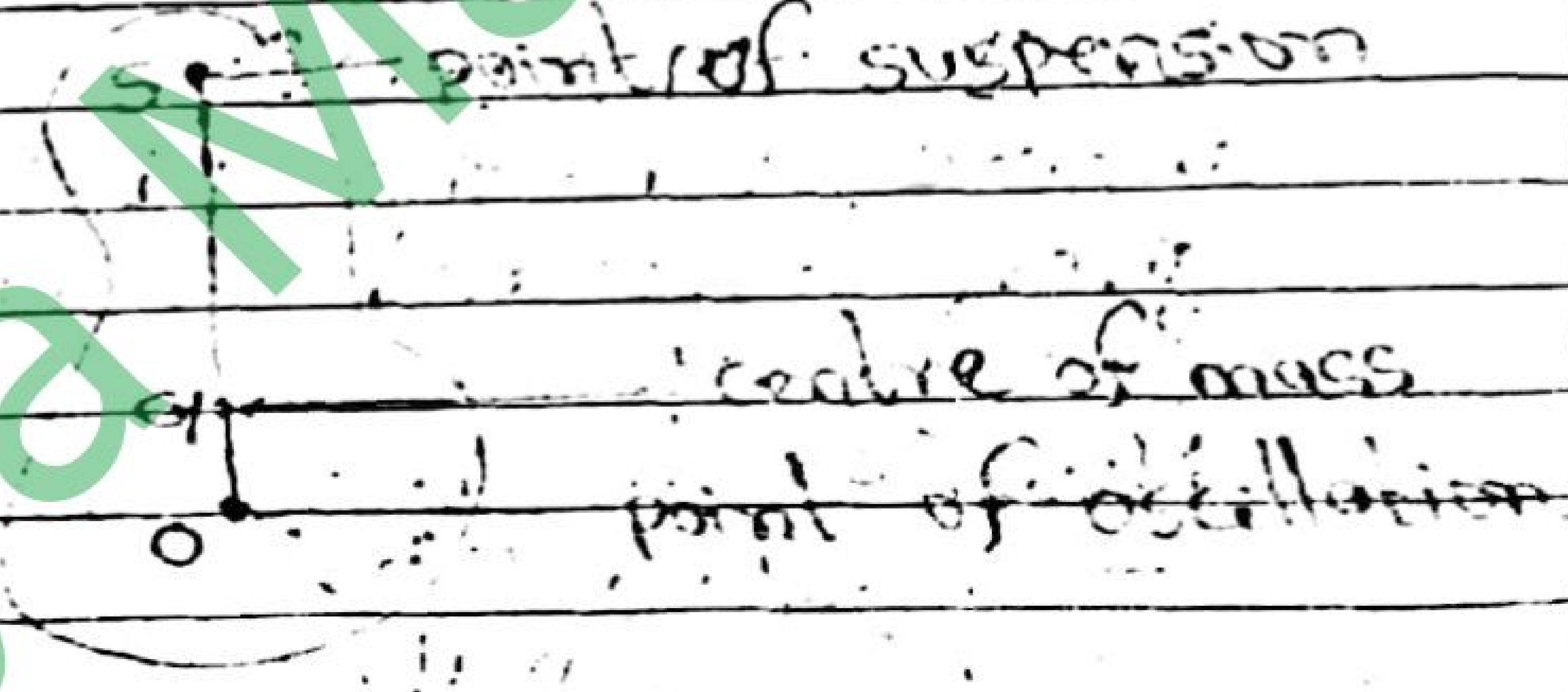
Where, $L = \frac{K^2}{l} + l$ is equivalent length of simple pendulum having equal time period of compound pendulum.

Point of oscillation in compound pendulum

We know in compound pendulum

$L = \left(\frac{K^2}{l} + l\right)$ where l be the equivalent length of simple pendulum having time period equal to that of given compound pendulum.

So, a point lying below by additional distance K^2/l^2 from centre of mass of given compound pendulum is called point of oscillation in compound pendulum. The figure below shows that point of oscillation (O).



Q→ Interchangeability of point of suspension is (s) and point of oscillation (O).

⇒ Solution,

Let T be the time period of a given compound pendulum when its point of suspension 's' and point of oscillation 'o'.

$$T = 2\pi\sqrt{\frac{\frac{K^2}{l} + l}{g}}$$

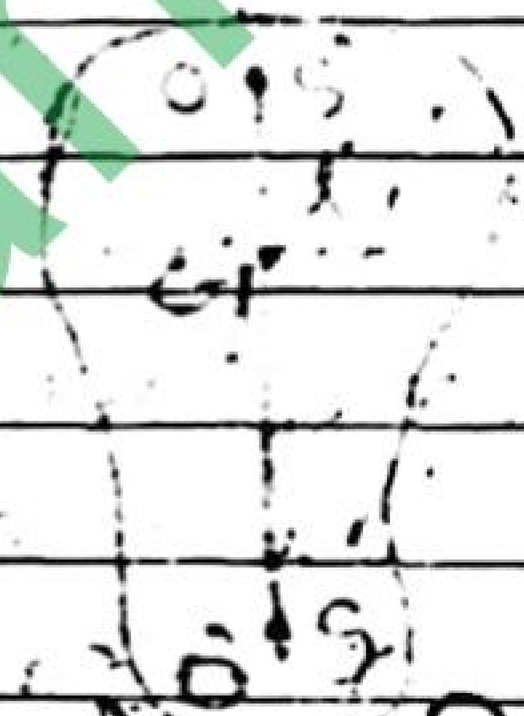
Or, $T = 2\pi\sqrt{\frac{l'+l}{g}} \longrightarrow (1)$

where $l' = \frac{K^2}{l}$

or, $K^2 = ll'$

Let T' be the new time period when point of oscillation is made new point of suspension as shown in figure.

$T' = 2\pi\sqrt{\frac{\frac{K^2}{l'}+l'}{g}}$



or $T' = 2\pi\sqrt{\frac{l+l'}{g}}$

This shows that the point of suspension and point of oscillation are interchangeable.

Maximum and minimum time period of compound pendulum

We know that

$T = 2\pi\sqrt{\frac{K^2/l + l}{g}}$

or, $T = 2\pi\sqrt{\frac{K^2+l^2}{gl}}$

or, $T = 2\pi\sqrt{\frac{(K-l)^2+2Kl}{gl}} \longrightarrow (1)$

<u>For minimum time period</u>

$K - l = 0 \quad K = l$ and then

$$T_{min} = 2\pi\sqrt{\frac{2K}{g}}$$

<u>For maximum time period</u>

$l = 0$ and then

$$T_{max} = \infty$$

This shows that compound pendulum has infinite time period when point of suspension coincides with centre of mass this means the compound pendulum does not oscillate rather rotate.

ADVANTAGES of Compound pendulum

1. Unlike the ideal simple pendulum, a compound pendulum is easily realizable in actual practice.
2. It oscillates as a whole and there is bob and the string in the case of simple pendulum.
3. The length to be measured is clearly defined.
4. On accounts of its large mass, and hence a large moment of inertia, it continues to oscillate for a larger time, thus enabling the time for a large no. of oscillations to be noted an its time period calculated more accurately.

Torsional pendulum

It consists a metalic wire whose one end is rigidly fixed and another end is fastened with solid disc or cylinder and suspended vertically as shown in figure.

When the disc is rotated along horizontal plane by angle θ the wire gets twisted by angle θ and create restoring torque τ given by

$$\tau = -C\theta \longrightarrow (1)$$

Where C is a twisting constant of the wire is given by

$$C = \frac{\pi \eta r^4}{2L} \longrightarrow (2)$$

Where, η is modulus of rigidity, r is radius and L is length of the wire.

If this restoring torque produces angular accleration 'α' and moment of inertia of the disc 'I' Then

$$\tau = I\alpha$$

$$\tau = I\frac{d^2\theta}{dt^2} \longrightarrow (3)$$

Equating eqn. (1) and (3), we get

$$I\frac{d^2\theta}{dt^2} = -C\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\left(\frac{C}{I}\right)\theta \longrightarrow (4)$$

This shows that angular accleration is directly proportional to twisting angle and negative sign indicates that it is bound towards mean position. So, motion of torsional pendulum is SHM.

Comparing this eq^n (4) with the general form $\frac{d^2\theta}{dt^2} = -\omega^2\theta$, we get

$$\omega = \sqrt{\frac{C}{I}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{C}{I}}$$

$$T = 2\pi\sqrt{\frac{I}{C}} \longrightarrow (5)$$

This gives the time period of torsonal ~~osc~~ pendulum.

For practical,

Squaring both side in eq^n (5), we get

$$T^2 = (2\pi)^2\frac{I}{C}$$

$$\text{or, } C = \frac{4\pi^2 I}{T^2}$$

$$\text{or, } \frac{\pi\eta r^4}{2L} = \frac{4\pi^2 I}{T^2}$$

$$\therefore \eta = \frac{8\pi I L}{T^2 r^4} \longrightarrow (6)$$

Using this relation modulus of rigidity of given wire can be found experimentally.

Helmholtz Resonator

It is a frequency specific device consisting of acoustic cavity contained within rigid walls connected to exterior by an opening called neck as shown in figure.

When air mass inside the neck in pushed inward, the air pressure inside the cavity increases due to compression. The increased pressure

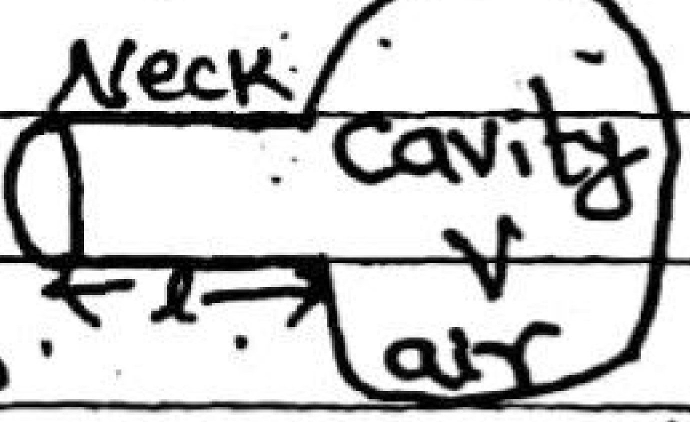


Fig: Helmholtz resonator

pused the air mass in the neck outward due to which air pressure inside the cavity decreases due to expansion. The reduced pressure inside the cavity again pulls the air mass in the neck inward. In this way, air mass in the neck moves to and fro like mass-spring system.

If l be the length of neck and a be its cross section area then air mass inside the neck becomes

$$m = la\rho$$

where ρ be the density of air.

If x be the inward displacement of air mass in the neck then decrease in volume is given by

$$dV = ax$$

If K be the bulk modulus of air then,

$$K = -\frac{dP}{(dV/V)}$$

$$\text{or, } dP = -K\left(\frac{dV}{V}\right)$$

$$\text{or, } dP = -K\left(\frac{ax}{V}\right) \longrightarrow (1)$$

This gives excess outward pressure on air mass inside the neck. So, force on air mass

$$F = dP.a$$

$$\text{or, } F = -\frac{K.a^2x}{V} \longrightarrow (2)$$

If this force produces accleration 'a' on air mass, then,

$F = ma$

or, $F = m\dfrac{d^2x}{dt^2}$

or, $F = la\rho\dfrac{d^2x}{dt^2} \longrightarrow (3)$

Equating (2) and (3), we get

$$la\rho\frac{d^2x}{dt^2} = \frac{-Ka^2x}{V}$$

or, $\dfrac{d^2x}{dt^2} = -\left(\dfrac{Ka}{l\rho V}\right)x \longrightarrow (4)$

This is in the form of $\dfrac{d^2x}{dt^2} = -\omega^2 x \longrightarrow (5)$

So, moment of air mass in the neck is SHM

Comparing (4) and (5), we get

$$\omega = \sqrt{\frac{Ka}{l\rho V}}$$

But $\sqrt{\dfrac{K}{\rho}} = c$ (speed of sound)

$\therefore \omega = c\sqrt{\dfrac{a}{lV}}$

or, $\dfrac{2\pi}{T} = c\sqrt{\dfrac{a}{lV}}$

or, $T = \dfrac{2\pi}{c}\sqrt{\dfrac{lV}{a}} \longrightarrow (6)$

This gives time period of helmholtz resonator.
Also,

$$f = \frac{1}{T} = \frac{r}{2\pi}\sqrt{\frac{\alpha}{lV}} \longrightarrow (7)$$

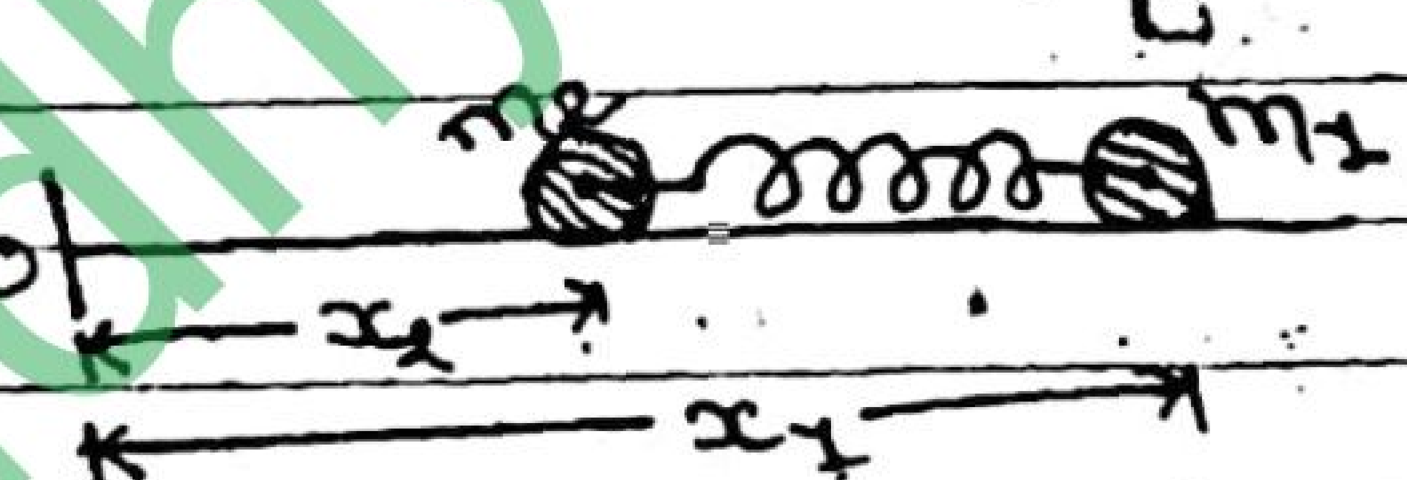
This gives the frequency of helmholtz resonator.

Two body harmonic oscillator

It consists of two separate body connected at the ends of a spring. When normal length length of the spring is disturbed, the system executes oscillation as SHM.

Consider a spring of normal length l_0 and m_1 and m_2 are the masses attached at both ends of the spring. c be the spring constant.

At any instant x_1 and x_2 be the distances of masses m_1 and m_2 from arbitrary origin as shown in figure.



If at any instant x is the change in normal length of spring then

$$x = (x_1 - x_2) - l_0 \Rightarrow (x_1 - x_2) = x + l_0$$

Here, x can be zero (neither elongation nor Contraction), positive (elongation) and negative (contraction) creating restoring force.

For perticular change in normal length by x, the force on m_1 due to m_2 can be expressed as,

$F_{12} = -cx$

$$m_1 \frac{d^2x_1}{dt^2} = -cx \longrightarrow (1)$$

Similarly, force on m_2 due to m_1 becomes,

$F_{21} = cx$

$$m_2 \frac{d^2x_2}{dt^2} = cx \longrightarrow (2)$$

Multiply eq^n (1) by m_2 and eq^n (2) by m_1 and substracting (2) from (1), we get

$$m_1m_2 \frac{d^2}{dt^2}(x_1 - x_2) = -c(m_1 + m_2)x$$

$$\text{or, } \frac{d^2(x_1 - x_2)}{dt^2} = -\frac{m_1 + m_2}{m_1m_2} cx$$

$$\text{or, } \frac{d^2(x + l_0)}{dt^2} = \frac{1}{\mu} cx$$

Where $\mu = \frac{m_1m_2}{m_1 + m_2}$ is reduced mass of the system.

$$\frac{d^2x}{dt^2} = \frac{c}{\mu} x \longrightarrow (3)$$

[Since l_0 is constant]

Clearly accleration of the system is directly proportional to change in length of the spring and -ve sign sign indicates that it always opposite the change.
Hence, oscillation of two body harmonic oscillator as SHM. we know, accleration equation for SHM in general form is given by

$$\frac{d^2x}{dt^2} = -\omega^2 x \longrightarrow (4)$$

Comparing eq^n (3) and (4), we get.

$$\omega = \sqrt{\frac{c}{\mu}}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{\frac{c}{\mu}}$$

$$\text{or, } T = 2\pi\sqrt{\frac{\mu}{c}}$$

This gives time period of two body harmonic oscillator. Thus the two body harmonic oscillator is reduced to single body harmonic oscillator. by the concept of reduced mass.

An ideal harmonic oscillator is that which indefinitely oscillates with constant amplitude having total energy in the form of kinetic and potential as constant. But in real practice when a body oscillates there exist some resistive forces like air resistance, viscous force, frictional force etc. which cause dissiption of energy of oscillating body and eventually amplitude of oscillating body continuously decreases.

The oscillatory motion of a body in which amplitude continuously decreases due to action of resistive force or damping force is called damped oscillation and the body is called damped harmonic oscillator.

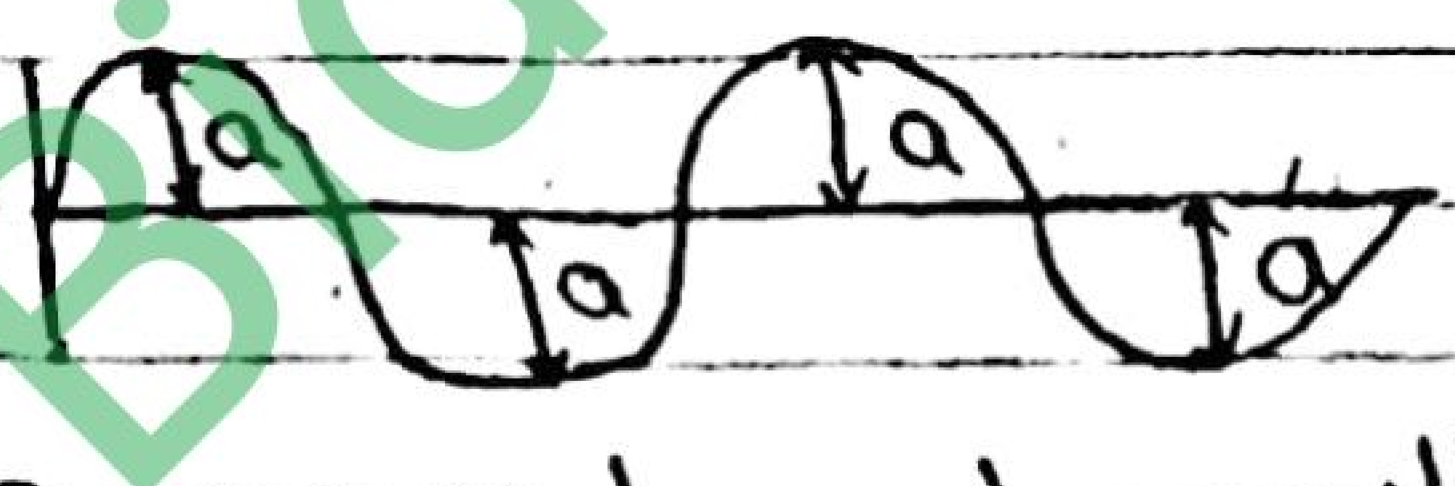


Fig. I : undamped oscillation

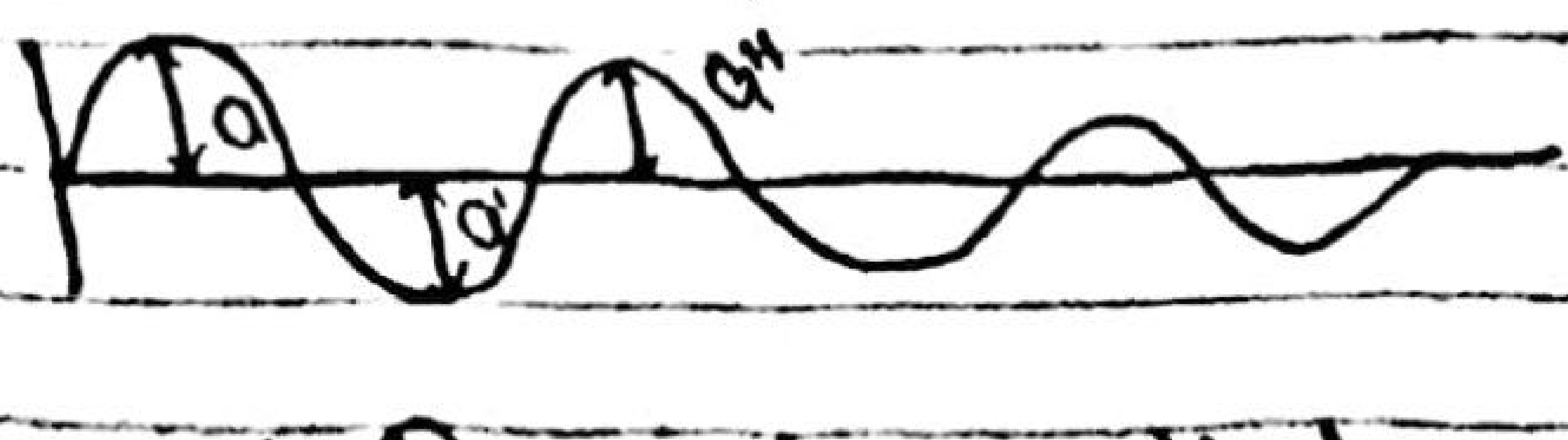


Fig. 2 : Damped oscillation

The damping force acting on oscillating body is directly proportional to instantaneous velocity of oscillating body and it always tends to ~~opposite~~ oppose the velocity. If $\frac{dx}{dt}$ is the instantaneous velocity of oscillating body for instantaneous displacement x then damping force can be written as

$$F_{damping} = -\gamma \frac{dx}{dt}$$

Where γ is damping constant.
Similarly, restoring force on the system is given by

$$F_{restoring} = -Cx$$

Where C is force constant of the oscillating system.
Hence, total force is

$$F = F_{damping} + F_{restoring}$$

$$\text{or, } F = -\gamma \frac{dx}{dt} - Cx$$

If $\frac{d^2x}{dt^2}$ is instantaneous accleration and m be the mass of oscillating body then, from Newton's 2nd law of motion

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - cx$$

or, $$\frac{d^2x}{dt^2} = -\frac{r}{m}\frac{dx}{dt} - \frac{c}{m}x$$

or, $$\frac{d^2x}{dt^2} = -\frac{1}{\tau}\frac{dx}{dt} - \omega_0^2 x$$

or, $$\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} + \omega_0^2 x = 0 \longrightarrow (1)$$

Where $\tau = \frac{m}{r}$ is relaxation time.

$\omega_0 = \sqrt{\frac{c}{m}}$ is natural angular frequency of oscillating body.

Usually eqn (1) can be expressed in more general form as

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = 0 \longrightarrow (2)$$

Where $\frac{1}{\tau} = 2k$ is effective damping constant.

This is 2nd order differential form and let its non-zero trial solution be in the form

$$x = Ae^{\alpha t}$$

$$\therefore \frac{dx}{dt} = \alpha A e^{\alpha t}$$

or, $\frac{d^2x}{dt^2} = \alpha A e^{\alpha t}$

putting this in eqn. (2), we get

$$\alpha^2 A e^{\alpha t} + 2K\alpha A e^{\alpha t} + \omega_0^2 A e^{\alpha t} = 0$$

or, $Ae^{\alpha t}(\alpha^2 + 2K\alpha + \omega_0^2) = 0$

or, $\alpha^2 + 2K\alpha + \omega_0^2 = 0$

This is quadratic equation in α.

So, $\alpha = \frac{-2K \pm \sqrt{4K^2 - 4\omega_0^2}}{2}$

or, $\alpha = -K \pm \sqrt{K^2 - \omega_0^2}$

$\therefore \alpha_1 = -K + \sqrt{K^2 - \omega_0^2}$

$\alpha_2 = -K - \sqrt{K^2 - \omega_0^2}$

So, $x_1 = A_1 e^{\alpha_1 t}$ and $x_2 = A_2 e^{\alpha_2 t}$

or, $x_1 = A_1 e^{(-K + \sqrt{K^2 - \omega_0^2})t}$ and

$x_2 = A_2 e^{(-K - \sqrt{K^2 - \omega_0^2})t}$

So, the general solution of eq^n (2) is of the form

$$x = x_1 + x_2$$

$$= A_1 e^{(-K+\sqrt{K^2-\omega_0^2})t} + A_2 e^{(K-\sqrt{K^2-\omega_0^2})t}$$

$$\therefore x = e^{-Kt}\left(A_1 e^{(\sqrt{K^2-\omega_0^2})t} + A_2 e^{(\sqrt{K^2-\omega_0^2})t}\right)$$

Case-I : Over damped

When $K > \omega_0$ the expression $\sqrt{K^2-\omega_0^2}$ is real and the oscillation decay experimentally being aperodic and dead beat.

Case-II : Under damped

When $K < \omega_0$ then expression $\sqrt{K^2-\omega_0^2}$ becomes imagginary and oscillation decay slowly i.e. amplitude decrease gradually.

Case-III : Critical damped

When $K = \omega_0$ then expression $\sqrt{K^2-\omega_0^2} = 0$ and $x = e^{-Kt}(A_1 + A_2)$. This situation is critical damped.