checked by 22/08/2080

Date: 2077
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Unit-2 Linear and angular Momentum

* Conservation of linear momentum:

The total quantity of motion contained in a body is called linear momentum. It is denoted by $\vec{p}$ and defined as the product of mass & velocity of the body. For a body having mass m and velocity $\vec{v}$. Its linear momentum $\vec{p}$ in vector form is given by

$$\vec{p} = m\vec{v}$$

In terms of magnitude only

$$p = mv$$

SI unit of linear momentum is $kgms^{-1}$ or NS and dimensional formula is $[MLT^{-1}]$.

* Law of conservation of linear momentum:

It states that "The total linear momentum of a system of particles remains constant, provided no external force acts on the system.

If $m_1, m_2 \ldots m_n$ be the particles in a system and $u_1, u_2 \ldots u_n$ be their respective then in absence of external force.

$$m_1u_1 + m_2u_2 + \ldots m_nu_n = \text{Constant}$$

$$\sum_{i=1}^{n} m_iu_i = \text{constant}$$

Proof:

(By using Newton second law of motion)

If a system has linear momentum p and applied force F causes change in linear momentum by dp in time dt then from Newton 2nd law of motion.

$$\frac{dp}{dt} = F$$

For validity of law of conservation of linear momentum force on the system must be zero

$$F = 0$$

then, $\frac{dp}{dt} = 0$

$$dp = 0$$

$$\boxed{\therefore p = \text{Constant}}$$

This proves law of conservation of linear momentum.

* Centre of mass: A point in a body or system of bodies where entire mass can be suppose to be located so that when the line of action of applied force passes through that point, the system of bodies gets linearly accelerated is called centre of mass.

If $m_1, m_2 \ldots m_n$ be the point masses situated radius vectors $\vec{r}_1, \vec{r}_2 \ldots \vec{r}_n$ from origin O. Then, radius vector of centre of mass from the same point can be mathematically expressed as:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

Where,

$M = \sum_{i=1}^{n} m_i$ is the total mass of system.

** If $m_1 = m_2 = \dots = m_n = m$ then,

$$\vec{r}_{cm} = \frac{m\vec{r}_1 + m\vec{r}_2 + \dots m\vec{r}_n}{m + m + \dots m}$$

$$= \frac{m(\vec{r}_1 + \vec{r}_2 + \vec{r}_3 \dots + \vec{r}_n)}{nm}$$

$$r_{cm} = \frac{\vec{r}_1 + \vec{r}_2 + \dots + \vec{r}_n}{n}$$

** For two particles system

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

If centre of mass coincides with the orbitary origion 'o' then

$$\vec{r}_{cm} = 0$$

$$\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = 0$$

$$\therefore m_1\vec{r}_1 = -m_2\vec{r}_2$$

In terms of magnitude only.

$$\boxed{\therefore m_1 r_1 = -m_2 r_2}$$

* Velocity of centre of mass:

We have,

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots m_n\vec{r}_n}{m_1 + m_2 + \ldots m_n}$$

or,
$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots m_n\vec{r}_n}{M}$$

Diff w.r.to time, we get

$$\frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots m_n\frac{d\vec{r}_n}{dt}\right)$$

or,
$$\vec{V}_{cm} = \frac{1}{M}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots m_n\vec{v}_n\right) \longrightarrow (I)$$

Eqn. (I) gives velocity of centre of mass.

Further,

$$M\vec{V}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots m_n\vec{v}_n$$

$$\vec{P}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots m_n\vec{v}_n \longrightarrow (II)$$

Eqn. (II) gives the linear momentum of centre of mass.

* Frame of reference:

The method of locating a point in space is called frame of reference. In general, there are two types of frame of reference normely.

(I) Labouratory frame of reference (L - frame)

(II) Centre of mass frame of reference (CM - frame)

(1) Labouratory frame of reference (L-frame) :-
The frame of reference in which a point in space is rigidly fixed as origin from where all other points are specified is called Labouratory frame

(2) Centre of mass frame of reference :-
The frame of reference in which centre of mass of a system under consideration is chosen as origin and from where all other point in space are specified is called centre of mass frame of reference (CM-frame).

* Collision :- An events in which interacting particles of a system exchange force for small interval of time by direct-indirect physical contact is called collision. On the basis of certain conservation laws collision can be broadly distinguished into two types :-
(I) Elastic collision :- The total linear momentum and kinetic energy of colliding system remain conserved / constant is known as elastic collision. Here, linear momentum and kinetic energy of individual member may change.
In general elastic collision occurs among microparticles of atomic and sub-atomic size. During elastic collision dissipative forms of energy like sound, heat, etc are not produced.

(II) <u>Inelastic Collision</u> :- The total linear momentum of colliding system remains conserved but not kinetic energy.

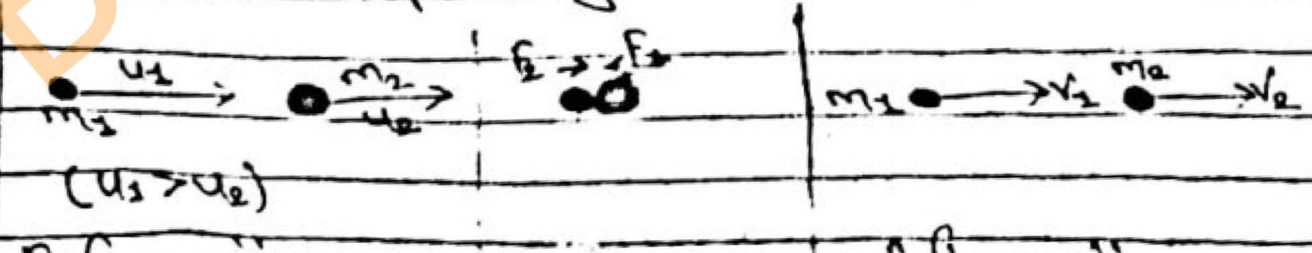
In general inelastic collision occurs among macroparticles. During inelastic collision dissipative forms of energy like heat sound etc are produced.

* <u>Perfectly inelastic Collision</u> :

An inelastic collision in which bodies stick together and move with common velocity after collision is called perfectly inelastic collision. Here, large fraction of kinetic energy gets dissipated.

* <u>Elastic collision in one-dimension (Two body system)</u>

Consider two bodies of masses m_1 and m_2 moving initially with velocities u_1 and u_2 along a line collide elastically for small interval of time so that they move now with velocities v_1 and v_2 respectively.



Before collision   During collision   After collision

from defination of elastic collision,
Total linear momentum before collision is equal to total linear momentum after collision.

i.e. $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$

$m_1(u_1 - v_1) = m_2(v_2 - u_2) \longrightarrow (I)$

Also,
Total kinetic energy before collision is equal to total kinetic energy after collision.

$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

$\Rightarrow m_1u_1^2 + m_2u_2^2 = m_1v_1^2 + m_2v_2^2$

$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2)$

$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2) \longrightarrow (II)$

from eq^n. (I) and (II), we get

$u_1 + v_1 = u_2 + v_2$

$u_1 - u_2 = v_2 - v_1 \longrightarrow (III)$

This shows that velocity of approach before collision is equal to the velocity of recede after collision.

Final velocities of elastic collision in terms of initial conditions:

For elastic collision,

$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \longrightarrow (I)$

Also,

$u_1 - u_2 = v_2 - v_1$

$v_2 = u_1 - u_2 + v_1 \longrightarrow (II)$

putting this value of v_2 from (II) in eq^n. (I), we get

$m_1u_1 + m_2u_2 = m_1v_1 + m_2u_1 - m_2u_2 + m_2v_1$

$\Rightarrow (m_1u_1 - m_2u_1) + 2m_2u_2 = (m_1+m_2)v_1$

$$\Rightarrow v_1 = \frac{(m_1-m_2)u_1}{m_1+m_2} + \frac{2m_2\,u_2}{m_1+m_2} \longrightarrow (A)$$

Similarly,

$$v_2 = \frac{2m_1}{(m_1+m_2)}u_1 + \frac{m_2-m_1}{m_1+m_2}u_2 \longrightarrow (B)$$

** Special cases:

Case-I : If colliding bodies have equal masses i.e. $m_1 = m_2 = m$, then

$v_1 = u_2$

$v_2 = u_1$

It shows that two bodies exchange their velocities.

Case-II : If colliding bodies have equal mass. i.e. $m_1 = m_2 = m$ and second body is intially at rest i.e. $u_2 = 0$,

$v_1 = 0, v_2 = u_1$

Case-III : If second body (target) is very massive i.e. $m_2 >> m_1$ and intially at rest i.e. $u_2 = 0$ then,

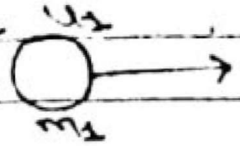
$v_1 = -u_1$ (rebounds with same velocities) and

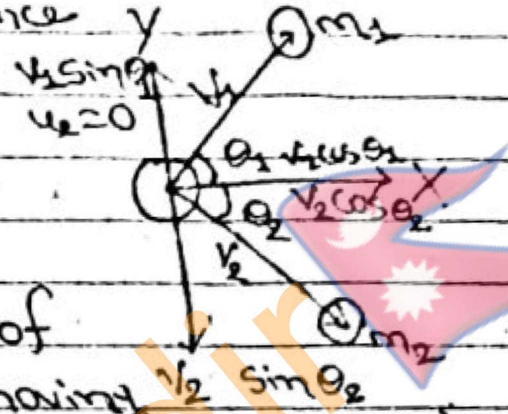
$v_2 \approx 0$ (still at rest)

Case-IV : If first body (Projectile) is very mass- i.e. ($m_1 >> m_2$) and second body is intially at rest, i.e. $u_2 = 0$ then $v_1 = u_1$ and $v_2 = 2u_1$

Elastic Collision in two Dimension (in L-frame)

If the origin of the reference frame is rigidly fixed at laboratory then the frame of reference is called L-frame.





Consider two particles of masses m_1 and m_2. m_1 is moving with velocity u_1 towards m_2 which is at rest. After collision m_1 moves with velocity v_1 making an angle θ_1 with x-axis which is called scattering angle. m_2 moves with velocity v_2 making angle θ_2 with x-axis which is called recoil angle. This shows that in figure.

Now,

Applying the law of conservation of linear momentum along given direction, we have

Along x-direction,

$$m_1u_1 + 0 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2$$

$$m_1u_1 = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 \longrightarrow (I)$$

Along y-direction,

$$0 = m_1v_1\sin\theta_1 - m_2v_2\sin\theta_2$$

$$m_1v_1\sin\theta_1 = m_2v_2\sin\theta_2 \longrightarrow (II)$$

Also, from conservation of kinetic energy, we have

$$\frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$m_1u_1^2 = m_1v_1^2 + m_2v_2^2 \longrightarrow (III)$$

To make relation solving, $m_1 = m_2 = m$ and say θ_1 is known then, eq^n (I), (II) and (III) becomes

$u_1 = v_1\cos\theta_1 + v_2\cos\theta_2 \longrightarrow (IV)$

$v_1\sin\theta_1 = v_2\sin\theta_2 \longrightarrow (V)$

$u_1^2 = v_1^2 + v_2^2 \longrightarrow (VI)$

Now, using (IV) in (VI), we get

$(v_1\cos\theta_1 + v_2\cos\theta_2)^2 = v_1^2 + v_2^2$

or, $v_1^2\cos^2\theta_1 + v_2^2\cos^2\theta_2 + 2v_1\cos\theta_1\cdot v_2\cos\theta_2 = v_1^2 + v_2^2$

or, $2v_1\cos\theta_1\cdot v_2\cos\theta_2 = v_1^2(1-\cos^2\theta_1) + v_2^2(1-\cos^2\theta_2)$

or, $2v_1\cos\theta_1\cdot v_2\cos\theta_2 = v_1^2\sin^2\theta_1 + v_2^2\sin^2\theta_2$

or, $2v_1\cos\theta_1\cdot v_2\cos\theta_2 = 2v_1^2\sin^2\theta_1$ (using (V))

or, $v_2\cos\theta_1\cdot\cos\theta_2 = v_1\sin^2\theta_1$

or, $v_2\cos\theta_1\cdot\cos\theta_2 = v_1(1-\cos^2\theta_1)$

or, $v_2\cos\theta_1\cdot\cos\theta_2 = v_1 - v_1\cos^2\theta_1$

or, $v_2\cos\theta_1\cdot\cos\theta_2 + v_1\cos^2\theta_1 = v_1$

or, $\cos\theta_1(v_2\cos\theta_2 + v_1\cos\theta_1) = v_1$

or, $u_1\cos\theta_1 = v_1$

$\boxed{\therefore v_1 = u_1\cos\theta_1} \longrightarrow (VII)$

Using eq^n (VII) in eq^n (VI), we have

$u_1^2 = u_1^2\cos^2\theta_1 + v_2^2$

$u_1^2(1-\cos^2\theta_1) = v_2^2$

$u_1^2\sin^2\theta_1 = v_2^2$

$\boxed{\therefore v_2 = u_1\sin\theta_1} \longrightarrow (VIII)$

* Inelastic Collision in one dimension

In inelastic collision conservation of linear momentum, remains true. so, $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$.

But kinetic energy doesn't get conserved. so

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \neq \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

for perfectly inelastic collision

$$m_1u_1 + m_2u_2 = (m_1+m_2)v$$

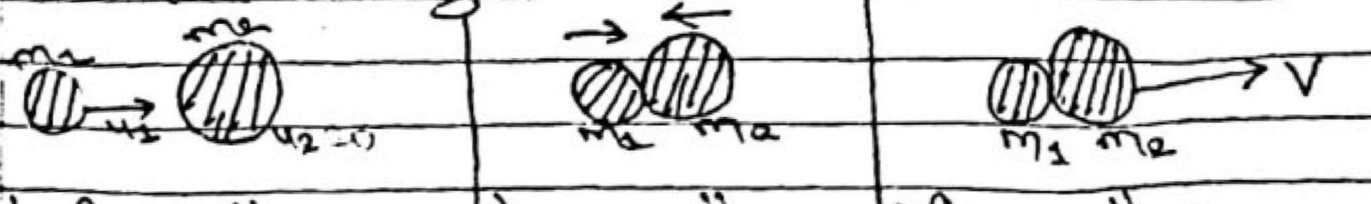
[since after collision two bodies stick together and move with common velocity]

Also,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \neq \frac{1}{4}(m_1+m_2)v^2$$

* Loss in kinetic energy during perfectly inelastic collision :-

Proof: Let us consider a body having mass m_1 and moving with velocity u_1 collides ~~initially~~ with another body of mass m_2 intially rest so, that both the bodies coalesce and move with common velocity v.



before collision	during collision	After collision

Now,
Applying law of conservation of linear momentum
$m_1u_1 + 0 = (m_1+m_2)v$

$v = \frac{m_1u_1}{m_1+m_2} \longrightarrow (i)$

Total K.E before collision $(K.E_i) = \frac{1}{2}m_1u_1^2 + 0$

$= \frac{1}{2}m_1u_1^2$

Total K.E. after collision $(K.E_f) = \frac{1}{2}(m_1+m_2)v^2$

Then,

$\frac{K.E_i}{K.E_f} = \frac{\frac{1}{2}m_1u_1^2}{\frac{1}{2}(m_1+m_2)v^2}$

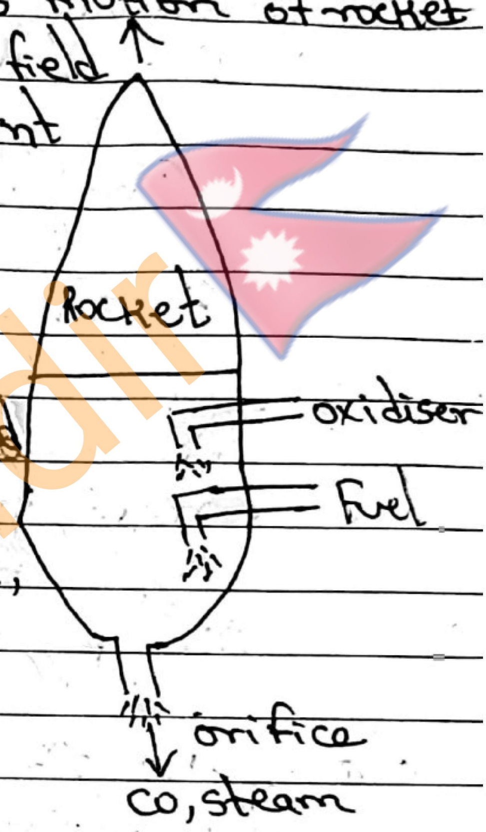
$= \frac{m_1u_1^2}{(m_1+m_2)\left(\frac{m_1u_1}{m_1+m_2}\right)^2}$

$\therefore \frac{K.E_i}{K.E_f} = \frac{m_1+m_2}{m_1}$

Here $(m_1+m_2) > m_1 \implies K.E_i > K.E_f$
Which shows that a kinetic energy decreases when perfectly inelastic collision occurs

Rocket (System of variable mass):

A rocket is a device which gets propelled against gravitational field due to upthrust created by burnt gases (fuel) released through orifice (tail end). The fuel itself carries its own oxidizer like gun powder. The thrust (force) provided by burnt gases released through tail end downward direction, is so massive that, the rocket body gets accelerated upward.



Theory :- Let M be the instantaneous mass of rocket after time 't' from its launch and $\frac{dM}{dt}$ be the rate of decrease of mass of the rocket due to exhaustion of burnt fuel. Let $\vec{v}$ be the instantaneous velocity of rocket (upward) from L-frame and $-\vec{v}'$ be the velocity of exhaust gas (downward) relative to rocket, we know,

Velocity of exhaust gas relative to rocket

$$\vec{V}_{gr} = \vec{V}_g - \vec{V}_r$$

$$\text{or,} \quad -\vec{v}' = \vec{V}_g - \vec{V}$$

$$\vec{V}_g = \vec{V} - \vec{v}' \longrightarrow (I)$$

This gives velocity of mass relative to E-frame. so downward force on gas

$$\vec{F} = -\frac{dM}{dt}(\vec{V}-\vec{v})$$

So, instantaneous force on rocket,

$$\vec{F} = \frac{dM}{dt}(\vec{V}-\vec{v}) \longrightarrow (II) \text{ (upward)}$$

This upward force changes the linear momentum of the rocket. As well as overcomes weight of rocket,

$$\text{So, } \vec{F} = \frac{d}{dt}(M\vec{V}) + Mg \longrightarrow (III)$$

** Case-I

Neglecting weight of rocket,

$$\vec{F} = \frac{d}{dt}(M\vec{V})$$

$$\text{or, } \frac{dM}{dt}(\vec{V}-\vec{v}) = M\frac{d\vec{V}}{dt} + \vec{V}\frac{dM}{dt}$$

$$\text{or, } \vec{V}\frac{dM}{dt} - \vec{v}\frac{dM}{dt} = M\frac{d\vec{V}}{dt} + \vec{V}\frac{dM}{dt}$$

$$\text{or, } M\frac{d\vec{V}}{dt} = -\vec{v}\frac{dM}{dt}$$

Taking magnitude only

$$M\,dV = -v\,dM$$

$$dV = -v\frac{dM}{M} \longrightarrow \text{(iv)}$$

Now integrating (iv)

$$\int dV = -v\int \frac{dM}{M}$$

or, $V = -v\ln M + C \longrightarrow \text{(v)}$

where, c is integrating constant

Now, imposing initial condition,

i.e. For $t=0$, $M = M_0$ and $V = V_0$ then,

$V_0 = -v\ln M_0 + C$

$\therefore C = V_0 + v\ln M_0$

Substituting this value of c in (iv), we have

$V = -v\ln M + V_0 + v\ln M_0$

$$V = V_0 + v\ln\frac{M_0}{M} \longrightarrow \text{(vi)}$$

$$\left[\because \ln a - \ln b = \ln\frac{a}{b}\right]$$

This gives velocity of rocket at any intermediate instant.

If $\frac{dM}{dt} = \beta$ then $M = M_0 - \beta t$

$$\therefore V = V_0 + v\ln\left(\frac{M_0}{M_0 - \beta t}\right)$$

$$\text{or, } V = V_0 + v\ln\left(\frac{1}{1 - \frac{\beta t}{M_0}}\right)$$

$$\therefore V = V_0 + v\ln\left(\frac{1}{1-\alpha t}\right) \quad \left[\because \alpha = \frac{\beta}{M_0}\right]$$

[rate of decrease in moves per unit initial mass]

$$V = V_0 + v\ln(1-\alpha t)^{-1}$$

or, $$\boxed{V = V_0 - v\ln(1-\alpha t)} \longrightarrow (VII)$$

** Case-II :

Taking weight of rocket under consideration,

$$\frac{dM}{dt}(V-v) = \frac{d}{dt}(MV) + (Mg)$$

or, $$V\frac{dM}{dt} - v\frac{dM}{dt} = M\frac{dV}{dt} + V\frac{dM}{dt} + Mg$$

or, $$M\frac{dV}{dt} = -v\frac{dM}{dt} - Mg$$

or, $$dV = -v\frac{dM}{M} - g\,dt \longrightarrow (VIII)$$

Integrating (VIII),

$$\int dV = -v\int\frac{dM}{M} - g\int dt$$

$$V = -v\ln M - gt + c \longrightarrow (IX)$$

where c is integrating constant

Now imposing initial condition, we get

for, $t=0$, $V=V_0$, $M=M_0$

$\therefore V_0 = -v\ln M_0 + c$

$\therefore c = V_0 + v\ln M_0 \longrightarrow (X)$

substituting the value of c from (X) to (IX), we get

$V = -v\ln M - gt + V_0 + v\ln M_0$

$\therefore \boxed{V = V_0 + v\ln\left(\frac{M_0}{M}\right) - gt} \longrightarrow (XI)$

Which is the velocity of a rocket at any intermediate instant after its launch with with consideration.

Q→7 Starting from Newton's 2nd law of motion deduce $v^2 = u^2 + 2as$ where the symbols have their usual meaning. [2070]

⇒ Solution,

We have, from Newton's 2nd law of motion

$f = ma = m\left(\frac{v-u}{dt}\right)$

$\Rightarrow a = \frac{v-u}{t}$

The average velocity of a body $= \left(\frac{v+u}{2}\right)$

Now,

The distance traveled in time 't' is given by

$s = \left(\frac{v+u}{2}\right)t = \left(\frac{v+u}{2}\right)\left(\frac{v+u}{a}\right)$

$\therefore 2as = v^2 - u^2 \Rightarrow v^2 = u^2 + 2as$

* Significance of C.M frame of reference

C.M. frame of reference makes it particularly useful in solving many a problem which are difficult to solve in the lab frame of reference. Indeed, it has become almost customary to deal with all collision in nuclear physics in this frame of reference alone.