Maxwell's Electromagnetic Equations

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Maxwell's Equation

Maxwell's equations are set of four differential equation that describe the behaviour of electromagnetic field in any medium or Vaccum. The Maxwell's equations are,

(I) $\nabla \cdot \vec{D} = \rho$ (Gauss law in electrostatics)

(II) $\nabla \cdot \vec{B} = 0$ (Gauss law in magnetostatics)

(III) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law of electromagnetic induction)

(IV) $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (Ampere circuital law)

Here,
$\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$

Equation of continuity

From the basic definition of current,

$$I = \frac{\partial q}{\partial t}$$

$$\text{or, } \int_s \vec{J}\, \vec{ds} = \frac{\partial}{\partial t}\left(\int_v \rho\, dv\right) \quad [\because I = \int_s \vec{J}\, ds \;\&\; q = \int_v \rho\, dv$$

$$\text{or, } \int_s \vec{J}\, \vec{ds} = \int_v \frac{\partial \rho}{\partial t}\, dv$$

or, $\int_v (\nabla \cdot \vec{J}) dv = \int_v \frac{\partial \rho}{\partial t} dv$

When the charge leaves the surface charge density 'ρ' decreases but current density ($\vec{J}$) increases. So we introduce a negative sign in above equation as

$$\int_v (\nabla \cdot \vec{J}) dv = -\int_v \frac{\partial \rho}{\partial t} dv$$

or, $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

$$\boxed{\therefore \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This gives the equation of the continuity.

TU
Q. 11 - **Proof of Maxwell's fourth equation ($\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$)**

Mathematical form of Ampere circuital law is

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

or, $\oint \mu_0 \vec{H} \cdot \vec{dl} = \mu_0 \int_S \vec{J} \cdot \vec{ds}$ $\left[\because \vec{B} = \mu_0 \vec{H} \text{ \& } I = \int_S \vec{J} \, \vec{ds} \right]$

or, $\oint \vec{H} \, \vec{dl} = \int_S \vec{J} \, \vec{ds}$

or, $\int (\nabla \times \vec{H}) \vec{ds} = \int_S \vec{J} \, \vec{ds}$

$\therefore \nabla \times \vec{H} = \vec{J}$

This equation holds true only when the electric field doesn't vary with time. So, for varying electric field some modification is needed. This modification was done by Maxwell's. So for varying electric field,

$\nabla \times \vec{H} = \vec{J} + \vec{J'} \longrightarrow (1)$

Where $\vec{J'}$ is a modification factor.

Taking divergence on both sides,

$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J'}$

$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{J'}$

or, $\nabla \cdot \vec{J'} = -\nabla \vec{J}$

or, $\nabla \cdot \vec{J'} = \frac{\partial \rho}{\partial t}$ $\left[\because \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \right]$

or, $\nabla \cdot \vec{J'} = \frac{\partial}{\partial t} \nabla \cdot \vec{D}$ $\left[\because \rho = \nabla \cdot \vec{D} \right]$

or, $\nabla \cdot \vec{J}' = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$

or, $\vec{J}' = \frac{\partial \vec{D}}{\partial t}$

Now, eqn. (1) becomes

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Which is the required Maxwell's fourth equation.

Displacement Current

If there is time variation of electric field in any region, a new kind of current is produced there. Such current produced due to time ~~period~~ variation of electric field and hence electric displacement is called displacement current.

The magnitude of displacement current is given by.

$$I_d = A\frac{\partial D}{\partial t}$$

The significance of displacement current is that with introduction of it, Ampere circuital law can be

$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 A$ 1/15

modified for both time varying and non varying electric field region.

Propagation equation fo electromagnetic field

(Q→ Write down Maxwell's equation in free space use the, and show that the electromagnetic field propagates there with velocity of light 3×10^8 m/s)

⇒ Maxwell's equation in free space are

(I) $\nabla \cdot \vec{D} = 0$

(II) $\nabla \cdot \vec{B} = 0$

(III) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(IV) $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

where,

$\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$

Taking Maxwell's 3rd equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad [\because \vec{B} = \mu_0 \vec{H}]$$

Taking curl on both sides

$\nabla \times (\nabla \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \vec{H})$

or, $\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \vec{H})$

or, $0 - \nabla^2 \vec{E} = -\mu_0 \frac{\partial}{\partial t} (\nabla \times \vec{H})$

[$\because \nabla \cdot \vec{D} = 0$ means $\nabla \cdot \vec{E} = 0$]

or, $\nabla^2 \vec{E} = \mu_0 \frac{\partial}{\partial t} \left(\frac{\partial \vec{D}}{\partial t} \right)$

[$\because$ Using Maxwell's 4th eqn]

or, $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow (1)$

Similarly, if we take fourth Maxwell's equation and proceed, we get

$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \longrightarrow (2)$

Both eqn (1) and (2) are in the form of

$\nabla^2 y = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$

So comparing them and we get

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$

or, $v = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$

or, $V = \sqrt{\frac{1}{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}}$

or, $V = 3 \times 10^8$ m/s

It shows that the electromagnetic field in free space propagates with velocity 3×10^8 m/s.

(5)

Proof of Ampere's law from Maxwell's equation

We have, from Maxwell's 4th eqn.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

or, $\nabla \times \left(\frac{\vec{B}}{\mu_0}\right) = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

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If the electric field doesnot vary in the time, then

$$\frac{\partial \vec{D}}{\partial t} = 0$$

So, $\nabla \times \vec{B} = \vec{J} \mu_0$

Taking surface integral both sides.

$$\int (\nabla \times \vec{B}) \cdot \vec{ds} = \mu_0 \int \vec{J} \cdot \vec{ds}$$

or, $\int (\nabla \times \vec{B}) \cdot \vec{ds} = \mu_0 I$

$\therefore \oint \vec{B} \cdot \vec{dl} = \mu_0 I$

[$\because$ using stoke theorem]

The above eqn is mathematical form of Ampere's law.

Q.11 Proof of Biot-Savart law from Maxwell's equation.

We have, from Maxwell's 4th eqn.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The electric field does not vary

$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A$

with time, $\frac{\partial \vec{D}}{\partial t} = 0$

So. $\nabla \times \vec{H} = \vec{J}$

or, $\nabla \times \left(\frac{\vec{B}}{\mu_0}\right) = \vec{J}$

or, $\nabla \times \vec{B} = \mu_0 \vec{J} \longrightarrow (1)$

If $\vec{A}$ is magnetic vector potential, then

$\vec{B} = \nabla \times \vec{A}$, eq^n. (1) becomes

$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J}$

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 A = \mu_0 \vec{J}$

According to Lorentz-Gauge, $\nabla \cdot \vec{A} = 0$

so,

$-\nabla^2 A = \mu_0 \vec{J}$

or, $\nabla^2 A = -\mu_0 \vec{J}$

This eq^n is in the form of poisson's equation, whose solution is

$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}}{r} dv$

Then,

Conducting medium
A medium is said to be conducting medium if both charge density (ρ) and current density (J) are non-zero. Metals are example of conducting medium.

Isotropic medium
A medium is said to be isotropic medium if the values of magnetic permeability (μ) and electric permittivity (ϵ) are constant in all directions. If a medium is isotropic and non conducting, the values of current density and charge density both are zero. For eg:- dielectric are isotropic non-conducting media.

Poynting vector ($\vec{S}$)
If $\vec{E}$ and $\vec{H}$ represents electric field and magnetic field vector then a vector obtained by their cross product is called poynting vector.

so,
$$\vec{S} = \vec{E} \times \vec{H}$$

or,
$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \quad [\because \vec{B} = \mu_0 \vec{H}]$$

The significance of poynting vector is that its direction gives the direction of propagation of electromagnetic field

IV Q. Discuss the behaviour of plane electromagnetic wave in isotropic dielectric or (non-conducting)

OR

Explain plane wave solution of Maxwell's equations

⇒ Ans:

Maxwell's equations for isotropic dielectric medium are.

(I) $\nabla \cdot \vec{D} = 0$

(II) $\nabla \cdot \vec{B} = 0$

(III) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(IV) $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

with $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu H$

Taking Maxwell's 3rd equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

or, $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$ $[\because \vec{B} = \mu H]$

Taking curl on both sides.

$\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$

or, $\cancel{\nabla(\nabla \cdot \vec{E})}^{0} - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left(\frac{\partial \vec{D}}{\partial t} \right)$

[$\because$ using 4th eqn.]

or, $-\nabla^2 \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2}$

or, $+\nabla^2 \vec{E} = \mu \frac{\partial^2}{\partial t^2} (\varepsilon \vec{E})$

or, $\nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow (1)$

In the similar manner, taking 4th eqn. and solving, we get

$\nabla^2 \vec{H} = \mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \longrightarrow (2)$

These two equations explain the propagation of electromagnetic field in isotropic dielectric media. The wave solutions of these two equations will be

$$E = E_0 e^{-i(\omega t - kx)} = E_0 e^{i(kx - \omega t)}$$

$$H = H_0 e^{i(kx - \omega t)}$$

Maxwell's eq^n in integral form

Maxwell's equation in integral form are as follows

(1) $\int_S \vec{D} \cdot \vec{ds} = q$

Proof :-

we have

$\nabla \cdot \vec{D} = \rho$

or, $\int_V (\vec{\nabla} \cdot \vec{D}) dV = \int_V \rho dV$

According to Gauge theorem,

$\int_V (\vec{\nabla} \cdot \vec{D}) = \int_S \vec{D} \cdot \vec{ds}$

So, $\int_S \vec{D} \cdot \vec{ds} = \int_V \rho dV$

$\therefore \int_S \vec{D} \vec{ds} = q$

(2) $\int_S \vec{B} \cdot \vec{ds} = 0$

Proof :-

we have

$\nabla \cdot \vec{B} = 0$

Integrating

$\int_V (\vec{\nabla} \cdot \vec{B})\, dv = 0$

Using Gauss theorem

$\int_S \vec{B} \cdot \vec{ds} = 0.$

(3): $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Integrating,

$\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot ds$

Using stoke theorem $\int_S (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = \int_C \vec{E} \cdot \vec{dl}$

so, $\oint_C \vec{E} \cdot \vec{dl} = -\frac{\partial}{\partial t} \int_C \vec{B} \cdot \vec{ds}$

(4) $\oint_C \vec{H} \cdot \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{ds}$

Proof :-

We have

$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Taking surface integration,

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \cdot \vec{ds}$$

Using stoke theorm,

$$\int (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \oint_c \vec{H} \cdot \vec{dl}$$

d, $\oint_c \vec{H} \cdot \vec{dl} = \oint_c \vec{H} \cdot \vec{dl}$

$$\oint_c \vec{H} \cdot \vec{dl} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \vec{ds}$$