Chapter-5
Wave Motion

Wave :
It is a mode of transfer energy from one end to another point due to periodic disturbance in medium. The disturbance in general, can be considered simple harmonic type.

On the basis of requirement of material medium or not wave can be distinguished into two types. They are (I) mechanical wave, (II) Electromagnetic wave.

1. Mechanical wave
The wave that essentially requires material medium for its propagation is called mechanical wave. In such wave, the propagation is affected by elastic and intertial property of the medium under consideration. Due to this reason the mechanical wave is also called elastic wave.

2. Electromagnetic wave
The wave that doesn't require material medium and can propagate even through vaccum is called electromagnetic wave. In such wave, there occurs periodic disturbance in electric field and magnetic field simultaneously. For eg: light, ultravilot

infraray, x-ray, γ-ray, radio wave etc.
On the basis of direction of energy transfer and direction of oscillation in medium wave can be distinguish in two types. They are (1) longitudinal wave (2) Transverse wave.

1. Longitudinal wave
The wave in which oscillation of medium partial takes place in parallel to line of energy propagation is called longitudinal wave. for eg. sound wave.
When longitudinal wave travels through a medium it brings periodic change in pressure of the midium due to formation of compression and rarefraction. so, for the propagation of longitudinal wave, the midium must offer bulk modulus or should be volume stress. As all three ~~satates~~ states solid, liquid and gas can tolerate volumestress the longitudinal wave can propogate through all these mediums.

2. Transverse wave.
The wave motion in which oscillation occurs being perpendicular to the line of energy propogation is called transverse

wave. Common example is wave along string under tension.

special cases → * All electromagnetic wave are transverse in nature. For mechanical transverse wave the medium most tolerate tangential or tenside stress so, mechanical transverse wave can propagate to solid only.]

* As free surface of liquid has capasity to tolerate tangential stress in small extent transverse wave can propagate. through free surface of liquid as well as. For eg: repulsion water surface.]

Relation between path difference and phase difference

We know, path difference is the separation between two oscillating particles lying on the line of propagation. It is denoted by x.

Similarly, phase difference is the difference in oscillation in angular form between two oscillating particles lying on the line of propagation. It is denoted by ϕ.

For path difference λ, phase difference

$= 2\pi$

For path difference 1, phase difference $= \frac{2\pi}{\lambda}$

For path difference x, phase difference $= \frac{2\pi}{\lambda} x = \phi$

$\therefore \phi = \frac{2\pi}{\lambda} x$

This gives the relation between phase difference and path difference between any two particles lying on line of propogation. Equivalently, $\phi = Kx$

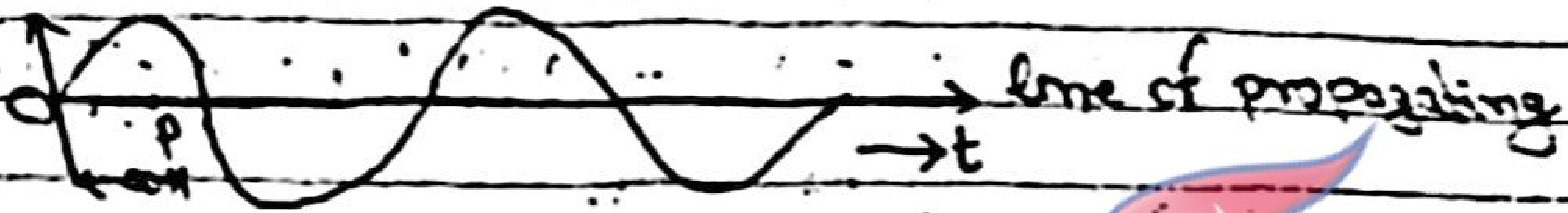
Where, $K = \frac{2\pi}{\lambda}$ is a constant called magnitude of wave vector.

Progressive wave.

The wave in which energy advances continuously along a line is called progressive wave.

Progressive wave in one dimension

Consider a wave travelling along positive x-axis with uniform speed v having wavelength λ, time period T, amplitude a, frequency f, angular frequency $\omega = 2\pi f$. Let 'o' be the arbitrary origin from where disturbance begins.



Now, the displacement eqⁿ at point O is given by
$y_0 = a\sin\omega t = a\sin 2\pi f t = a\sin 2\pi \frac{v}{\lambda} t$
for any particle P lying on line of propagation at distance 'x' right from origin 'O' can be expressed as

$$y = a\sin(\omega t - \phi)$$

$$\text{or, } y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$\therefore y = a\sin(\omega t - kx) \longrightarrow (1)$$

other general forms are

$$y = a\sin\left(2\pi\frac{t}{T} - \frac{2\pi}{\lambda}x\right)$$

$$\text{or, } y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \longrightarrow (2)$$

Also,

$$y = a\sin\left(2\pi\frac{v}{\lambda}t - \frac{2\pi}{\lambda}x\right)$$

$$\text{or, } y = a\sin\frac{2\pi}{\lambda}(vt - x) \longrightarrow (3)$$

General forms of progressive wave

General forms of progressive wave are

$$y = a\sin(\omega t - kx) = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

For the wave travelling along -ve direction the equations (i) becomes,

$$y = a\sin(\omega t + kx) = a\sin\left(\omega t + \frac{2\pi}{\lambda}x\right)$$

$$y = a\sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)$$

$$y = a\sin\frac{2\pi}{\lambda}(vt + x)$$

Wave eqn. insted of sine form also can be cosine form.

$$y = a\cos\left(\omega t - \frac{2\pi}{\lambda}x\right)$$

$$y = a\cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

$$y = a\cos\frac{2\pi}{\lambda}(vt - x)$$

Wave velocity and particle velocity

Wave velocity represents the velocity of energy and that remains constant for given wave in given medium. But particle velocity represents the oscillatory motion of medium particle lying on the line of propagation and this velocity varies periodically in simple harmonic manner.

We know, displacement of eq^n of oscillating particle is given by

$$y = a\sin\frac{2\pi}{\lambda}(vt - x) \longrightarrow (1)$$

where v be the wave velocity.

Now, particle velocity can found by diff. equation (1) with respect to time (t). so

$$u = \frac{dy}{dt} = \frac{d}{dt}\left(a\sin\frac{2\pi}{\lambda}(vt-x)\right)$$

$$\text{or, } u = a\frac{2\pi}{\lambda}v\cos\frac{2\pi}{\lambda}(vt-x)$$

$$\text{or, } u = \frac{2\pi}{\lambda}v\,a\cos\frac{2\pi}{\lambda}(vt-x) \longrightarrow (2)$$

Again diff. eq^n (1) w.r.to x, we get

$$\frac{dy}{dx} = -a\frac{2\pi}{\lambda}\cos\frac{2\pi}{\lambda}(vt-x)$$

or, $\frac{dy}{dx} = -\frac{2\pi}{\lambda} a\cos\frac{2\pi}{\lambda}(vt-x) \longrightarrow (3)$

Comparing eqn (2) and (3), we get

$$y = -v\frac{dy}{dx} \longrightarrow (4)$$

So, particle velocity = - wave velocity x slope of y vs x curve.

* The slope of y verses x curve is zero at extrime position so particle becomes zero there.
* The slope of y verses x curve is maximum at mean position so particle velocity ~~beom~~ becomes maximum there.

Progressive wave equation (In differential form)

We know, displacement equation for a progressive wave is given by

$$y = a\sin\frac{2\pi}{\lambda}(vt-x) \longrightarrow (1)$$

Differentiating eqn (1) w.r.to time (t), w

$$\frac{dy}{dt} = \frac{2\pi}{\lambda} av\cos\frac{2\pi}{\lambda}(vt-x)$$

Again,
Differentiating this w.r.to 't' we get

$$\frac{d^2y}{dt^2} = \frac{2\pi}{\lambda} \cdot \frac{2\pi}{\lambda} av.v - \sin\frac{2\pi}{\lambda}(vt - x)$$

or, $$\frac{d^2y}{dt^2} = -\frac{4\pi^2}{\lambda^2} av^2 \sin\frac{2\pi}{\lambda}(vt-x)$$

or, $$\frac{d^2y}{dt^2} = -\frac{4\pi^2}{\lambda^2} v^2 y \quad [\text{using } 1] \longrightarrow (2)$$

Now, Diff. (1) w.r.to x, we get

$$\frac{dy}{dx} = a.\frac{2\pi}{\lambda} \cos\frac{2\pi}{\lambda}(vt-x).$$

Also,

$$\frac{d^2y}{dx^2} = a.\frac{2\pi}{\lambda} \cdot \frac{2\pi}{\lambda}\left[-\sin\frac{2\pi}{\lambda}(vt-x)\right]$$

or, $$\frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} a \sin\frac{2\pi}{\lambda}(vt-x)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} y \longrightarrow (3)$$

Comparing eq^n. (2) and (3), we get.

$$\frac{d^2y}{dx^2} = \frac{1}{v^2} \frac{d^2y}{dt^2} \longrightarrow (4)$$

or,

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \longrightarrow (5)$$

Eqn (4) and (5) represents progressive wave in 2nd order differential form. in one dimension.

* In three dimensions,

$$\nabla^2 y = \frac{1}{v^2}\frac{d^2y}{dt^2}$$

where,

$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is laplacian operator.

$\frac{d^2y}{dx^2}$ = Curvature

$\frac{dy}{dx}$ = slope

Stationary/standing wave

Review

A wave that forms after the superposition of a ~~perfecte~~ progressive wave with its own reflection is called stationary or standing wave.

In a stationary wave, there exist certain points/positions where amplitude becomes zero and such points are called nodes. Similarly, there exist certain points where amplitude becomes maximum and such points are called antinodes.

General form of stationary wave eqn

Consider a progressive wave of amplitude a travelling along positive x-direction with constant velocity (v) having frequency (f), angular frequency (ω), wavelength λ that reflects and retracts its path forming a stationary wave.

Let y_1 be the displacement at any point due to ongoing progressive wave then, (left → right)

$y_1 = a\sin(\omega t - Kx)$

~~From principle of superposition the resultant displacement becomes;~~

$\cancel{y = y}$

Let y_2 be the displacement at same point due to reflected then, (right to left)

$y_2 = a\sin(\omega t + Kx)$

From principle of superposition, the resultant displacement becomes

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= a\sin(\omega t - Kx) + a\sin(\omega t + Kx) \\
&= a[\sin\omega t \times \cos Kx - \cos\omega t \times \sin Kx + \sin\omega t \times \cos Kx + \cos\omega t \times \sin Kx] \\
&= a\,2\sin\omega t\cos Kx \\
&= (2a\cos Kx)\sin\omega t \longrightarrow (1)
\end{aligned}
$$

Here, $2a\cos Kx = A \longrightarrow (2)$

Eq^n (2) represents resultant amplitude of stationary wave. it is clear that the resultant amplitude of stationary wave doesnot remain constant rather varies with position.

* For minima

The node is a position where resultant amplitude becomes zero. so,

$A = 0$

$2a\cos kx = 0$

or, $\cos kx = 0$

or, $\cos kx = \cos(2n+1)\frac{\pi}{2}$ where; $n = 0, 1, 2, 3 \ldots$

or, $kx = (2n+1)\frac{\pi}{2}$

or, $\frac{2\pi}{\lambda} x = (2n+1)\frac{\pi}{2}$

or, $x = \frac{(2n+1)}{4}\lambda$

* For maxima

The antinode is a position where resultant amplitude of standing wave becomes maximum.

i.e. A = maximum

$\cos kx = \pm 1$

or, $\cos kx = n\pi$

$Kx = n\pi$

$$x = \frac{n\pi}{\frac{2\pi}{\lambda}} = \frac{n\lambda}{2}$$

Expression for energy density of plane progressive wave

Consider a progressive wave travelling through a material medium with amplitude 'a', angular frequency $\omega = 2\pi f$ with constant speed v such that instantaneous displacement eqn. in simplest form can be expressed as

$$y = a\sin\omega t$$

So, instantaneous particle velocity can be expressed as

$$u = \frac{dy}{dt} = a\omega\cos\omega t$$

If m is the mass of vibrating layer of particles present in the medium then K.E. of the layer will be

$$K.E = \frac{1}{2}mu^2$$

$$\text{or, } K.E = \frac{1}{2}ma^2\omega^2\cos^2\omega t \longrightarrow (1)$$

We know, total energy of oscillating layer is the sum of K.E and P.E which is in fact equal to maximum K.E. So,

Totatal energy (E) = $K.E_{max}$

$$or,\ E = \frac{1}{2} m a^2 \omega^2$$

$[For\ K.E_{max}\ \cos\omega t = 1]$

Now energy density (u) is

$$= \frac{\text{Total energy}}{\text{volume}}$$

$$= \frac{E}{V} = \frac{\frac{1}{2} m a^2 \omega^2}{V}$$

$$u = \frac{1}{2} \rho a^2 \omega^2 \longrightarrow (2)$$

Where ρ be the density of vibrating layer of medium during propagation of wave.

Further,

$\omega = 2\pi f$ then,

$$u = \frac{1}{2} \rho a^2 4\pi^2 f^2$$

$$\therefore u = 2\rho a^2 \pi^2 f^2$$

Wave intensity

The wave energy per unit time per unit cross section propogating perpendicularity is called wave intensity. It is denoted by I

We have

Total energy $(E) = \frac{1}{2} m a^2 \omega^2$

In the time t, length of disturbed medium due to propogation of a wave is given by

$l = vt$

If A be the cross sectional area of the medium under distubance then volume of disturbed medium becomes

$V = Al = Avt$

$\therefore$ Mass of disturbed medium

$(m) = V\rho$

$= Avt\rho$

$\therefore E = \frac{1}{2} Avt\rho a^2 \omega^2$

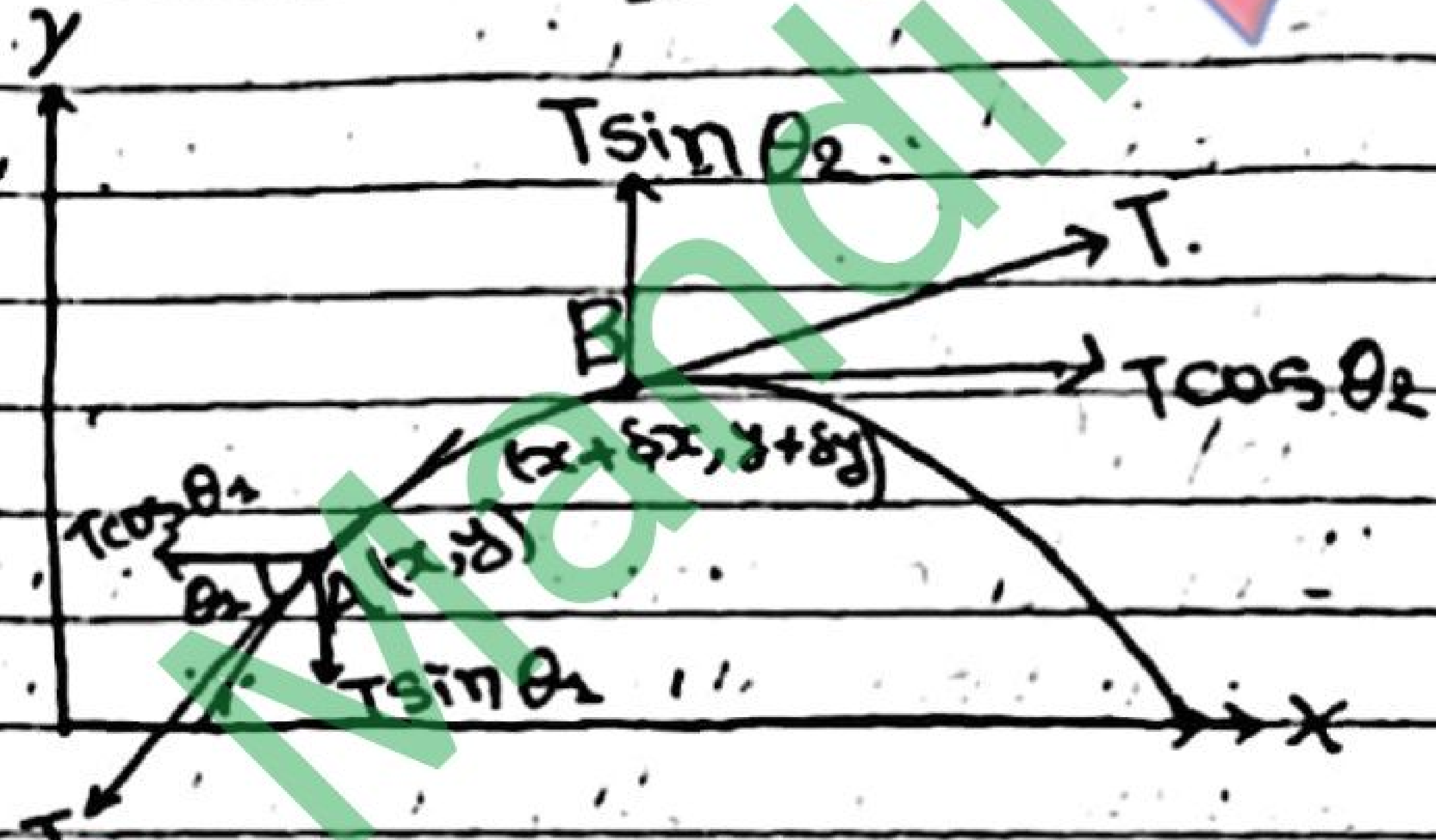
or, $\frac{E}{At} = \frac{1}{2} v\rho a^2 \omega^2$

138

$$\therefore \boxed{I = \frac{1}{2}\rho v a^2 \omega^2}$$

This gives wave intensity

Velocity of Transverse wave in a stretched string



Let us consider AT and BT be the two tangents at A and B making an angle θ_1 and θ_2 respectively with x-axis. Let tension force at A and B be T. From the figure, the resultant force on element AB along Y-axis

$F = T\sin\theta_2 - T\sin\theta_1$

If θ_1 and θ_2 are very small then,

$\sin\theta_1 \approx \tan\theta_1$ and

$\sin\theta_2 \approx \tan\theta_2$

$\therefore$ $T\tan\theta_2 - T\tan\theta_1$

$= T(\tan\theta_2 - \tan\theta_1)$

Since, $\tan\theta = \frac{dy}{dx}$

$\therefore F = T\left[\frac{d}{dx}(y+\delta y) - \frac{dy}{dx}\right]$

$= T\left[\frac{d}{dx}\left(y + \frac{dy}{dx}\delta x\right) - \frac{dy}{dx}\right]$

$\because \delta y = \frac{dy}{dx}\delta x$

$= T\left[\frac{dy}{dx} + \frac{d^2y}{dx^2}\delta x - \frac{dy}{dx}\right]$

$\therefore F = T\frac{d^2y}{dx^2} \longrightarrow (1)$

If m be the mass per unit length of the string then.
Mass of element $AB(\delta x)$, $\delta m = m\delta x$
So, force on element δx can be written
Therefore, $F = \delta m \frac{d^2y}{dt^2}$

$F = m\delta \frac{d^2y}{dt^2} \longrightarrow (2)$

Equating (1) and (2)

$T\frac{d^2y}{dx^2} = m\delta x\frac{d^2y}{dt^2}$

$$T\frac{d^2y}{dx^2} = m\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} = \frac{T d^2y}{m dx^2} \longrightarrow (3)$$

Comparing eqn. (3) with general wave eqn. $\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$ we get

$$v^2 = \frac{T}{m}$$

$$\therefore v = \sqrt{\frac{T}{m}} \longrightarrow (4)$$

Hence, eqn. (4) gives the velocity of Transverse wave along stretched string#

Next - Method

Let us consider wave travelling along the string A to B with velocity v. consider a small element CD of length Δl. T be then tension in a string acting along tangent at C and D. This small element CD forms an arc of circle having radius R ($OC = OD = R$) and $\angle COD = \theta + \theta = 2\theta$.

Since the string moves along a circular path, it should be under the action of centripetal force is given as

$$F = 2T\sin\theta$$

$$F = 2T\theta \quad (\because \sin\theta \simeq \theta)$$

we have,

$$2\theta = \frac{\overset{\frown}{arc}}{radius} = \frac{\overset{\frown}{CD}}{R} = \frac{\Delta l}{R}$$

If μ be the mass per unit length of string. So mass of small element CD

$$m = \mu \times \Delta l$$

Also, we have

Centripetal force is

$$F = \frac{mv^2}{R}$$

$$2T\theta = \frac{\mu \times \Delta l\, v^2}{R}$$

$$\frac{\Delta l}{R} T = \frac{\mu \Delta l v^2}{R}$$

$$T = \mu v^2$$

$v^2 = \frac{T}{\mu}$

$\therefore v = \sqrt{\frac{T}{\mu}}$

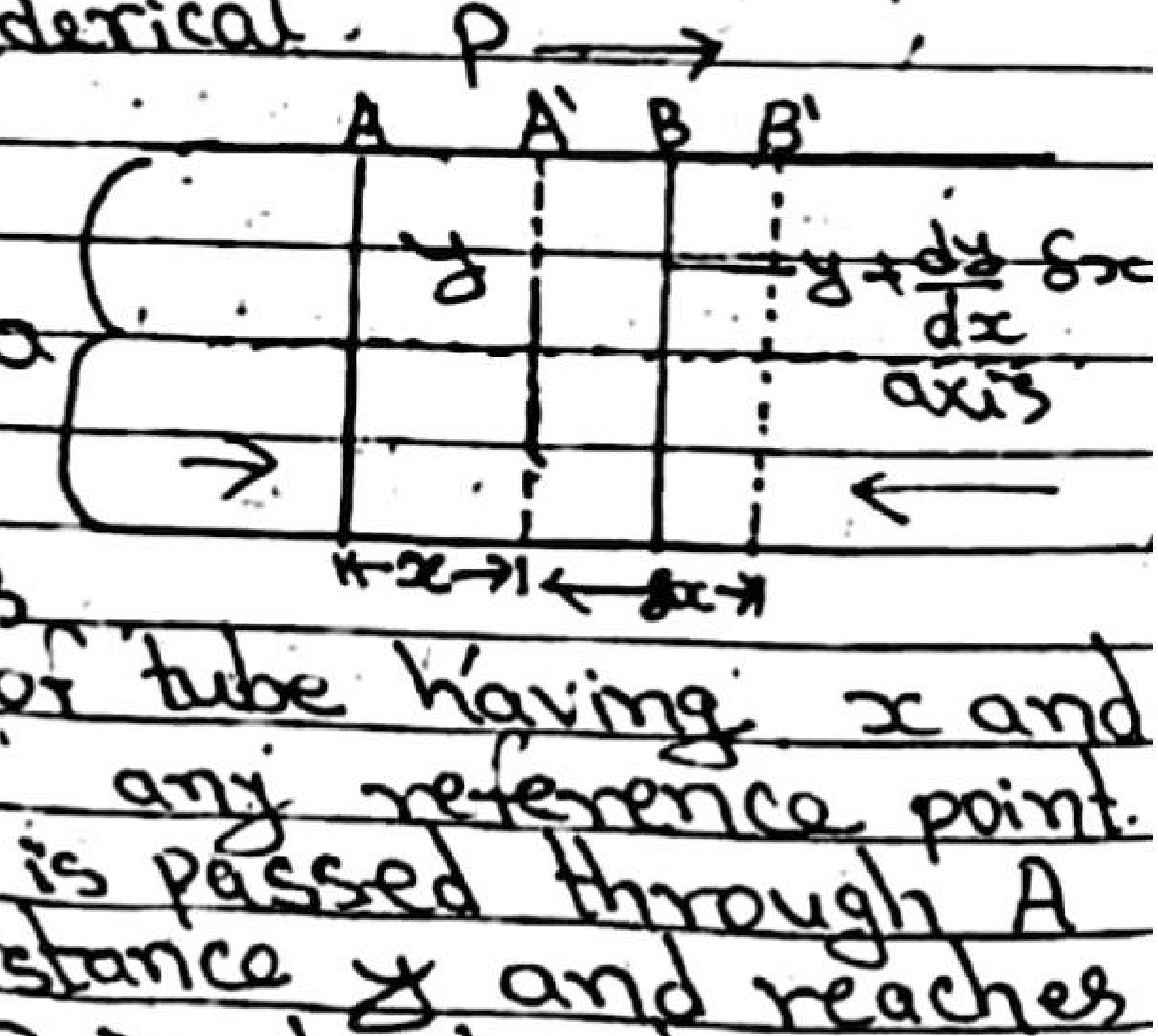
Then, the wave equation is

$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2}$

or, $\frac{d^2y}{dt^2} = \frac{T}{\mu} \frac{d^2y}{dx^2}$

Velocity of longitudinal waves in gases or liquid medium

Consider a cylinderical tube of uniform cross sectional area a, the wave travelling left to right. Here A and B are two planes



perp. to the axes of tube having x and δx distance from any reference point. So, sound wave is passed through A is displaced by distance y and reaches to A'. Similarly B is displaced by $y + \frac{dy}{dx}\delta x$ and reaches to B'

Now, the increase in length of slice AB is

$$y + \frac{dy}{dx}\delta x - y = \frac{dy}{dx}\delta x$$

Increase in volume of slice AB

$$= \frac{dy}{dx}\delta x\, a$$

So, the volumetric strain $= \frac{dy/dx\ \delta x . a}{\delta x . a}$

$$= \frac{dy}{dx}$$

The bulk modulus of elasticity is given as

$$K = -\frac{P}{dy/dx}$$

$$\therefore P = -K\frac{dy}{dx}$$

Where P is excess pressure at A, now similarly at A'. Similarly the excess pressure at B now at B' is

$$P + \frac{dP}{dx}\delta x$$

$\therefore$ The resultant pressure acting in slice AB is

$$P + \frac{dP}{dx}\delta x - P$$

$$= \frac{dP}{dx} \delta x$$

$$= \frac{d}{dx}\left(-K\frac{dy}{dx}\right)\delta x$$

$$= -K\frac{d^2y}{dx^2}\delta x$$

$$= K\frac{d^2y}{dx^2}\delta x$$

Here -ve sign is drop.

Hence, Resulant force acting in the slice AB is

$$F = K\frac{d^2y}{dx^2}\delta x . a \longrightarrow (1)$$

If $\frac{d^2y}{dt^2}$ is the accleration produced in the slice AB then the force acting in the slice AB is

$$F = \rho \, \delta x \, a \frac{d^2y}{dt^2} \longrightarrow (2)$$

Equating eqn (1) and (2), we get

$$\frac{d^2y}{dx^2} = \frac{1}{K/\rho}\frac{d^2y}{dt^2} \longrightarrow (3)$$

Comparing eqn. (3) with general eqn. of wave $\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$ we get

$$v = \sqrt{\frac{K}{\rho}}$$

Newton assumed that the sound wave travels in gaseous medium under isothermal condition. so he replaced bulk modulus of elasticity 'K' by isothermal elasticity of the gas 'P' which is equal to the pressure of gas

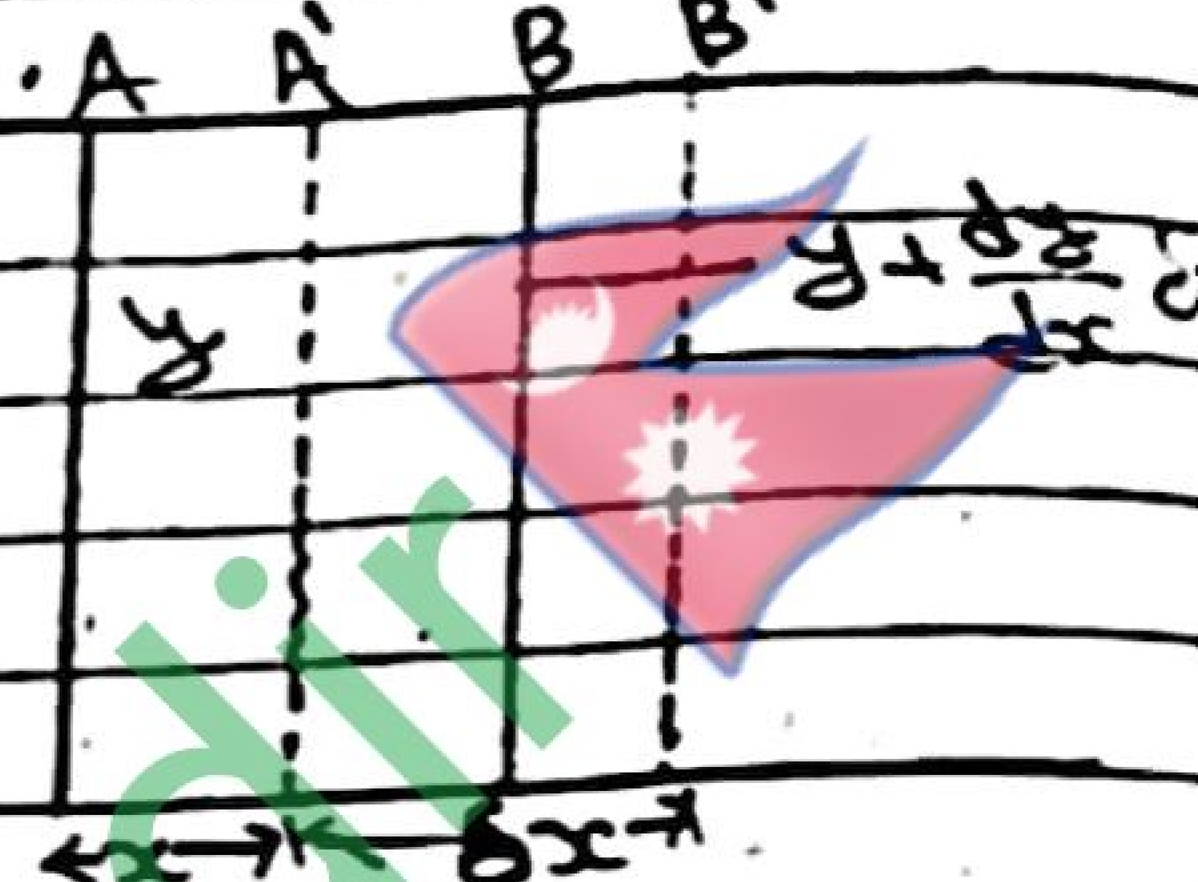
$$\therefore v = \sqrt{\frac{P}{\rho}} = 280 \text{ m/s}$$

This value is not good agreemental value. And then Laplace corrected this Newton's formula by assuming that sound wave travels in gaseous medium under adiabatic expansion condition. He replaced K by adiabatic elasticity of the gas i.e. $\gamma \times$ isothermal elasticity = adiabatic elasticity.

$$\therefore v = \sqrt{\frac{\gamma P}{\rho}} = 332 \text{ ms}^{-1}$$ ##

Velocity of Longitudinal wave in rod

Let us consider a cylindrical tube of cross sectional area a. A and B be the two planes perp. to the axis of rod having distance x and δx resp. from any reference point.



The change in length of slice AB is

$$y + \frac{dy}{dx}\delta x - y$$

$$= \frac{dy}{dx}\delta x$$

$$\text{Longitudinal strain} = \frac{\frac{dy}{dx}\delta x}{\delta x}$$

$$= \frac{dy}{dx}$$

Now, Young's modulus of elasticity is

$$Y = \frac{\text{Stress}}{\text{strain}} = \frac{F/a}{dy/dx}$$

$$\therefore F = aY\frac{dy}{dx}$$

Where F be the restoring force at A now at A'. Also the restoring force at B now at B', is

$$F + \frac{dF}{dx}\delta x$$

The resultant force in the slice AB is

$$F + \frac{dF}{dx}\delta x - F = \frac{d}{dx}\left(aY\frac{dy}{dx}\right)\delta x$$

$$= aY\frac{d^2y}{dx^2}\delta x$$

$$\therefore F = aY\frac{d^2y}{dx^2}\delta x \longrightarrow (1)$$

If $\frac{d^2y}{dt^2}$ be the accleration produced in the section AB then force acting inthis section is

$$F = \rho a \delta x \frac{d^2y}{dt^2} \longrightarrow (2)$$

Where ρ be the density of material of a rod.

Now, Equating eq^n (1) and (2), we get

$$\frac{d^2y}{dx^2} = \rho/Y \frac{d^2y}{dt^2} \longrightarrow (3)$$

Comparing eqn (3) with the general eqn of wave $\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2}$, we get

$$v^2 = \frac{Y}{\rho}$$

$$\therefore v = \sqrt{\frac{Y}{\rho}} \longrightarrow (4)$$

This eqn (4) gives the velocity of longitudinal wave in a rod.