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Differentiation of vectors

1. Define derivative of a vector function of a scalar variable. prove that the necessary and sufficient condition for the vector function $\vec{a}$ of a scalar variable 't' to have a constant magnitude $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$. A particle moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. Find the magnitude of the velocity at $t = \pi/3$.

[2056, 2070]

⇒ Solution,

(i) vector function:

A vector function $\vec{f}(t)$ is said to be differentiable or derivable for the value t_0 of t if the limit

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \text{ exists.}$$

If it exists, it is known as the derivative of $f(t)$ at $t = t_0$ and is denoted by $\frac{d}{dt}\{\vec{f}(t_0)\}$.

(I) Necessary condition:

Let $\vec{a}$ be a vector function of a scalar variable t. suppose $\vec{a}$ has a constant magnitude

$\therefore \vec{a} \cdot \vec{a} = a^2$; a = constant $\longrightarrow$ (1)

Differentiating both sides w.r.to t; we get

$$\frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt}(a^2)$$

or, $\vec{a}\cdot\frac{d\vec{a}}{dt}+\frac{d\vec{a}}{dt}\cdot\vec{a}=0$

or, $2\vec{a}\cdot\frac{d\vec{a}}{dt}=0$ $\left[\because \vec{a}\cdot\frac{d\vec{a}}{dt}=0\right]$

The sufficient condition

Let $\vec{a}$ be a vector function of scalar variable t that satisfy the relation

$\vec{a}\cdot\frac{d\vec{a}}{dt}=0$

To prove $\vec{a}$ has a constant magnitude

we know that

$\vec{a}\cdot\vec{a}=a^2=|\vec{a}||\vec{a}|$

Differentiating both sides w.r.to t, we get

$2\vec{a}\cdot\frac{d\vec{a}}{dt}=2|\vec{a}|\left|\frac{d\vec{a}}{dt}\right|$

$\therefore |\vec{a}|\frac{d}{dt}|\vec{a}|=0$

Since $|\vec{a}|\neq 0$, so $\frac{d}{dt}|\vec{a}|=0$

$\vec{a}$ is a constant vector

Hence, $\vec{a}$ has constant magnitude.

Next part:

The particle moves along the curve $x=2\sin 3t$, $y=2\cos 3t$, $z=8t$

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$\Rightarrow$

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Let $\vec{r}$ be the position vector of a particle at any time 't'. Then

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

or, $\vec{r} = 2\sin 3t\,\vec{i} + \cancel{6\sin}\ 2\cos 3t\,\vec{j} + 8t\,\vec{k}$

Differentiating both side with respect to 't'

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(2\sin 3t\,\vec{i} + 2\cos 3t\,\vec{j} + 8t\,\vec{k}\right)$$

or $\vec{V} = 6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k}$

where $\vec{v}$ is velocity of particle.

Now, at $t = \pi/3$

velocity $(\vec{v}) = 6\cos 3\frac{\pi}{3}\vec{i} - 6\sin 3\frac{\pi}{3}\vec{j} + 8\vec{k}$

$= 6\cos\pi\,\vec{i} - 6\sin\pi\,\vec{j} + 8\vec{k}$

$= -6\times 1\,\vec{i} - 6\times 0\,\vec{j} + 8\vec{k}$

$= -6\vec{i} + 8\vec{k}$

And the magnitude of velocity is

$v = \sqrt{(-6)^2 + 8^2} = \sqrt{36+64} = 10$ unit.

2. Show that the necessary and sufficient condition for the vector function $\vec{v}$ of scalar variable 't' to have a constant magnitude is $\vec{v}\cdot\frac{d\vec{v}}{dt} = 0$ [2054, 2055, 2059, 2067, 2068]

$\Rightarrow$ Solution;

Necessary condition

Let $\vec{v}$ is a vector function of a scalar variable 't' has a constant magnitude.

$\therefore \vec{v}.\vec{v} = v^2$; a constant $\longrightarrow$ (1)

Differentiating both sides w.r.to t, we get

$$\frac{d}{dt}(\vec{v}.\vec{v}) = \frac{d}{dt}(v^2)$$

$$\vec{v}\frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt}.\vec{v} = 0$$

$$\text{or, } 2\vec{v}\frac{d\vec{v}}{dt} = 0$$

$$\therefore \vec{v}.\frac{d\vec{v}}{dt} = 0$$

The sufficient condition

Let $\vec{v}$ be a vector function of scalar variable t satisfies the relation

$$\vec{v}.\frac{d\vec{v}}{dt} = 0$$

To prove $\vec{v}$ has a constant magnitude.

We know $\vec{v}.\vec{v} = v^2 = |\vec{v}||\vec{v}|$

Differentiating both sides w.r.to t, we get

$$2\vec{v}.\frac{d\vec{v}}{dt} = 2|\vec{v}|\frac{d|\vec{v}|}{dt} \quad \left[\because \vec{v}.\frac{d\vec{v}}{dt}\right]$$

$$\therefore |\vec{v}| \neq 0, \text{ so } \frac{d}{dt}(\vec{v}) = 0$$

$|\vec{v}|$ is constant vector
Hence, $\vec{v}$ has constant magnitude.

3. If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$ then show that $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$; where $\vec{a}$ and $\vec{b}$ are constant vectors. [2056]

⇒ Solution,
Given,
$\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t \longrightarrow (1)$

Differentiating both side with respect to 't'

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(+\vec{a}\cos\omega t + \vec{b}\sin\omega t)$$

or, $$\frac{d\vec{r}}{dt} = -\vec{a}\,\omega\sin\omega t + \vec{b}\,\omega\cos\omega t$$

or, $$\frac{d\vec{r}}{dt} = \omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

Again, differentiating both side w.r.to t

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}[-\omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)]$$

$$= [\omega(-\vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t)]$$

$$= -\omega^2(\vec{a}\cos\omega t + \vec{b}\sin\omega t)$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$$

Proved

5. A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the magnitude of accleration and velocity.
[2057, 2063, 2071, 2073]

⇒ Solution,

Given,

$x = 4\cos t,\ y = 4\sin t,\ z = 6t$

Suppose,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

or, $\vec{r} = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\,\vec{k}$

Differentiating with respect to t

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\,\vec{k}\right)$$

or, $\frac{d\vec{r}}{dt}$ = velocity $= -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$

when $t = \pi$, then

$$\frac{d\vec{r}}{dt} = -4\sin\pi\,\vec{i} + 4\cos\pi\,\vec{j} + 6\vec{k}$$

$$= -4 \times 0\,\vec{i} + 4 \times 1\,\vec{j} + 6\vec{k}$$

$$= 4\vec{j} + 6\vec{k}$$

∴ magnitude of ~~acc~~ velocity (V)

$= \sqrt{4^2 + 6^2}$

$= \sqrt{16 + 36}$

= 7 unit

Again,

Differentiating $\frac{d\vec{r}}{dt}$ w.r.to t

or, $\frac{d^2\vec{r}}{dt^2}$ = Accleration $= \frac{d}{dt}(-4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k})$

$= -4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 0$

$= -4\cos t\,\vec{i} - 4\sin t\,\vec{j}$

When $t = \pi$ then

$\frac{d^2\vec{r}}{dt^2} = -4\cos\pi\,\vec{i} - 4\sin\pi\,\vec{j}$

$= 4\vec{i}$

$\therefore$ Magnitude of accleration $a = \sqrt{4^2 + 0^2}$

$= 4$ unit.

5. A particle p is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$ Show that its accleration is $-\omega^2\vec{r}$.

$\Rightarrow$ solution, [2065]

Let particle p is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$

Then,

$x = r\cos\omega t$, and $y = r\sin\omega t$

so, $\vec{r} = r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j}$ $[\because \vec{r} = x\vec{i} + y\vec{j}]$

Now differentiating both side w.r.to t, we get

$\frac{d\vec{r}}{dt} = -r\omega\sin\omega t\,\vec{i} + r\omega\cos\omega t\,\vec{j}$

Also,

$$\frac{d^2\vec{r}}{dt^2} = -r\omega^2\cos\omega t\,\vec{i} - r\omega^2\sin\omega t\,\vec{j}$$

$$= -\omega^2(r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j})$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$$

Proved

6. If $\vec{r_1}$ and $\vec{r_2}$ be the differentiable vector functions, then prove that:

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$$

[BSc 2068 old]

⇒ Solution,

Let, $\vec{r} = \vec{r_1}\times\vec{r_2} \longrightarrow (1)$

Let $\Delta\vec{r_1}$, $\Delta\vec{r_2}$, $\Delta\vec{r}$ and Δt are small increments in $\vec{r_1}$, $\vec{r_2}$, $\vec{r}$ and t respectively, where t is scalar

$\therefore \vec{r} + \Delta\vec{r} = (\vec{r_1} + \Delta\vec{r_1})\times(\vec{r_2} + \Delta\vec{r_2})$

or, $\Delta\vec{r} = \vec{r_1}\times\vec{r_2} + \vec{r_1}\times\Delta\vec{r_2} + \Delta\vec{r_1}\times\vec{r_2} + \Delta\vec{r_1}\times\Delta\vec{r_2} - \vec{r_1}\times\vec{r_2}$

or, $\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\vec{r_1}\times\Delta\vec{r_2} + \Delta\vec{r_1}\times\vec{r_2} + \Delta\vec{r_1}\times\Delta\vec{r_2}}{\Delta t}$

When $\Delta t \to 0$

$$\lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t\to 0}\left(\frac{\vec{r_1}\times\Delta\vec{r_2} + \Delta\vec{r_1}\times\vec{r_2} + \Delta\vec{r_1} + \Delta\vec{r_2}}{\Delta t}\right)$$

or, $\dfrac{d\vec{r}}{dt} = r_1\times\dfrac{dr_2}{dt} + \dfrac{dr_1}{dt}\times r_2$

$\therefore \dfrac{d}{dt}(r_1\times r_2) = r_1\times\dfrac{dr_2}{dt} + \dfrac{dr_1}{dt}\times r_2$

Proved

7. Obtain the derivative of vector product of two differential vector functions. [2072]

⇒ See Q.N. 6.

8. Obtain the derivative of the vector triple product. [2071-new]

⇒ Solution,

Let r_1, r_2 and r_3 be three vectors, their vector product is given by

$$\vec{r} = \vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3) \longrightarrow (1)$$

Differentiating w.r.to t, we get

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3))$$

$$= \frac{d\vec{r}_1}{dt} \times (\vec{r}_2 \times \vec{r}_3) + r_1 \times \frac{d}{dt}(\vec{r}_2 \times \vec{r}_3)$$

$$= \frac{d\vec{r}_1}{dt} \times (\vec{r}_2 \times \vec{r}_3) + \vec{r}_1 \times \frac{d r_2}{dt} \times \vec{r}_3 + \vec{r}_1 \times \left(\vec{r}_2 \times \frac{d\vec{r}_3}{dt}\right)$$

9. What do you mean by constant vector? Evaluate $\frac{d}{dt}\left(\frac{\vec{r} - \vec{a}}{r^2 + a^2}\right)$ [2071].

⇒ Solution,

Constant vector:

A vector $\vec{a}$ is said to be constant vector if both magnitude and direction of $\vec{a}$ are constant. If $\vec{a}$ is constant then $\frac{d\vec{a}}{dt} = 0$

Let $\vec{v} = \frac{\vec{r}-\vec{a}}{r^2+a^2}$

Diff. both sides w.r. to t, we get

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}-\vec{a}}{r^2+a^2}\right)$$

$$= \frac{(r^2+a^2)\frac{d}{dt}(\vec{r}-\vec{a}) - (\vec{r}-\vec{a})\frac{d}{dt}(r^2+a^2)}{(r^2+a^2)^2}$$

$$= \frac{1}{r^2+a^2}\frac{d}{dt}(\vec{r}-\vec{a}) - \frac{2(\vec{r}-\vec{a})\vec{r}}{(r^2+a^2)^2}\cdot\frac{d\vec{r}}{dt}$$

10. If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, prove that

$\vec{r} \times \frac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$ [2070]

$\Rightarrow$ Solution,

Given,

$\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$

Diff. both sides w.r.to 't', we get

$$\frac{d\vec{r}}{dt} = -\omega\vec{a}\sin\omega t + \omega\vec{b}\cos\omega t$$

$$\frac{d\vec{r}}{dt} = \omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

Then,

$$\vec{r}\times\frac{d\vec{r}}{dt} = (\vec{a}\cos\omega t + \vec{b}\sin\omega t)\times\omega(-\vec{a}\sin\omega t + \vec{b}\cos\omega t)$$

$$= \omega\{(\vec{a}\times\vec{b})\cos^2\omega t + (\vec{b}\times(-\vec{a}))\sin^2\omega t\}$$

$$= \omega\{(\vec{a}\times\vec{b})\cos^2\omega t + (\vec{a}\times\vec{b})\sin^2\omega t\}$$

$$= \omega(\vec{a}\times\vec{b})(\cos^2\omega t + \sin^2\omega t)$$

$$= \omega(\vec{a}\times\vec{b})$$

$\therefore \vec{r}\times\dfrac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$ proved

11. If $\dfrac{d}{dt}\vec{a} = \vec{c}\times\vec{a}$, $\dfrac{d}{dt}\vec{b} = \vec{c}\times\vec{b}$ then show that

$\dfrac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$ [2057, 2060]

$\Rightarrow$ Solution,

$\dfrac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$, $\dfrac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$

Then

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \cancel{\frac{d}{dt}(\vec{c}\times\vec{a})}\ \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

$$= \vec{a}\times(\vec{c}\times\vec{b}) + (\vec{c}\times\vec{a})\times\vec{b}$$

$$= \vec{a}\times(\vec{c}\times\vec{b}) - \vec{b}\times(\vec{c}\times\vec{a})$$

$$= (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c} + (\vec{b}\cdot\vec{c})\vec{a}$$

$$\cancel{= (\vec{a}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}}$$

$$= (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$$

$$= \vec{c}\times(\vec{a}\times\vec{b})$$

$\therefore \dfrac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$

Proved

12. If $\vec{r}_1 = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and $\vec{r}_2 = t^2\vec{i} + t\vec{j} - (t-1)\vec{k}$, then find (i) $\frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2)$ and (ii) $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2)$. [2061, 2070]

$\Rightarrow$ Solution,

Let, $\vec{r}_1 = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$

and, $\vec{r}_2 = t^2\vec{i} + t\vec{j} - (t-1)\vec{k}$

$\cancel{(I)\ \frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = [}$

(I) $\vec{r}_1 \cdot \vec{r}_2 = \left[(2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}\right] \cdot \left[t^2\vec{i} + t\vec{j} - (t-1)\vec{k}\right]$

$= (2t+1)t^2 - t^2 t - 3t^3(t-1)$

$= 2t^3 + t^2 - t^3 - 3t^4 + 3t^3$

$= t^2 + 4t^3 - 3t^4$

$\therefore \frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = \frac{d}{dt}(t^2 + 4t^3 - 3t^4)$

$= 2t + 12t^2 - 12t^3$

$= 2t(1 + 6t - 6t^2)$

(II) $\vec{r}_1 \times \vec{r}_2 = \left[(2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}\right] \times \left[t^2\vec{i} + t\vec{j} - (t-1)\vec{k}\right]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ t^2 & t & -t+1 \end{vmatrix}$$

$= (t^3 - t^2 - 3t^4)\vec{i} + (3t^5 + 2t^2 - 2t + t - 1)\vec{j} + (2t^2 + t + t^4)$

$= (t^3 - t^2 - 3t^4)\vec{i} + (3t^5 + 2t^2 - t - 1)\vec{j} + (2t^2 + t + t^4)\vec{k}$

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$\therefore \frac{d}{dt}(\vec{r_1} \times \vec{r_2})$

$= \frac{d}{dt}\left[(t^3 - t^2 - 3t^4)\vec{i} + (3t^5 + 2t^2 - 2t - 1)\vec{j} + (2t^2 + t + t^4)\vec{k}\right]$

$= (3t^2 - 2t - 12t^3)\vec{i} + (15t^4 + 4t - 1)\vec{j} + (4t + 1 + 4t^3)\vec{k}$

13. If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$, then find

(i) $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ (ii) $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$

$\Rightarrow$ Solution, [2062, 2063, 2066, 2070, 2075]

Given,

$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$

Differentiating both sides w.respect to 't'

$\frac{d\vec{r}}{dt} = \frac{d}{dt}(a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k})$

$= -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$

Also

$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}(-a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k})$

Also, $= -a\cos t\,\vec{i} - a\sin t\,\vec{j}$.

Again,

$\frac{d^3\vec{r}}{dt^3} = \frac{d}{dt}(-a\cos t\,\vec{i} - a\sin t\,\vec{j})$

$= a\sin t\,\vec{i} - a\cos t\,\vec{j}$

(I) $\therefore \dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2} = (-a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k})$
$\times (-a\cos t\,\vec{i} - a\sin t\,\vec{j})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (a^2\tan\alpha\sin t)\vec{i} + (-a^2\tan\alpha\cos t)\vec{j} + a^2\vec{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^4(1+\tan^2\alpha)}$$

$$= a^2\sec\alpha$$

(II) Also,

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt} \cdot \left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3}\right)$$

So, $\dfrac{d^2\vec{r}}{dt^2} \times \dfrac{d^3\vec{r}}{dt^3} = (-a\cos t\,\vec{i} - a\sin t\,\vec{j}) \times$
$(a\sin t\,\vec{i} - a\cos t\,\vec{j})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

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$= (0+0)\vec{i} - (0+0)\vec{j} + (a^2\cos^2 t + a^2\sin^2 t)\vec{k}$

$= a^2(\sin^2 t + \cos^2 t)\vec{k}$

$= a^2\vec{k}$

$\therefore \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = (-a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}).(a^2\vec{k})$

$= 0+0+a^3\tan\alpha$

$= a^3\tan\alpha.$

14. If $\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{v} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$, find $\frac{d}{dt}(\vec{u}.\vec{v})$ and $\frac{d}{dt}(\vec{u}\times\vec{v})$ at $t=1$

[2073]

⇒ Solution,

Given,

$\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

$\vec{v} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$

$\therefore \vec{u}.\vec{v} = [t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}].[(2t-3)\vec{i} + \vec{j} - t\vec{k}]$

$= 2t^3 - 3t^2 - t - 2t^2 - t$

$= 2t^3 - 5t^2 - 2t$

Now,

$\frac{d}{dt}(\vec{u}.\vec{v}) = \frac{d}{dt}(2t^3 - 5t^2 - 2t)$

$= 6t^2 - 10t - 2$

At $t=1$ then

$\therefore \frac{d}{dt}(\vec{u}.\vec{v}) = 6 - 10 - 2 = -6.$

Again,

$$\vec{u}\times\vec{v} = (t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}) \times ((2t-3)\vec{i} + \vec{j} - t\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$$= (t^2 - 2t - 1)\vec{i} + (4t^2 + 2t - 6t - 3 + t^3)\vec{j} + (t^2 + 2t^2 - 3t)\vec{k}$$

$$= (t^2 - 2t - 1)\vec{i} + (t^3 + 4t^2 - 4t - 3)\vec{j} + (3t^2 - 3t)\vec{k}$$

$$\therefore \frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d}{dt}\left[(t^2-2t-1)\vec{i} + (t^3+4t^2-4t-3)\vec{j} + (3t^2-3t)\vec{k}\right]$$

$$= (2t-2)\vec{i} + (3t^2+8t-4)\vec{j} + (6t-3)\vec{k}$$

At $t = 1$ then,

$$\therefore \frac{d}{dt}(\vec{u}\times\vec{v}) = 0\vec{i} + 7\vec{j} + 3\vec{k}$$

$$= 7\vec{j} + 3\vec{k}.$$

15. For the space curve $x = 3t,\ y = 3t^2,\ z = 2t^3$, prove that

$$\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right] = 216 \quad [2072, 2075]$$

$\Rightarrow$ Solution,

Given,

$x = 3t,\ y = 3t^2,\ z = 2t^3$

Let $\vec{r} = (x, y, z)$

or, $\vec{r} = (3t, 3t^2, 2t^3)$ be any vector.

$$\frac{d\vec{r}}{dt} = (3, 6t, 6t^2)$$

$$\frac{d^2\vec{r}}{dt^2} = (0, 6, 12t)$$

$$\frac{d^3\vec{r}}{dt^3} = (0, 0, 12)$$

Now,

$$\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 3 & 6t & 6t^2 \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix}$$

$$= 3(72-0) + 6t(0-0) + 6t^2(0-0)$$

$$= 216 \text{ Proved}$$

19. If $\vec{r}_1 = 2t^2\vec{i} + 3(t-1)\vec{j} + 4t^2\vec{k}$ and $\vec{r}_2 = (t-1)\vec{i} + t^2\vec{j} + (t-2)\vec{k}$, show that $\int_0^2 (\vec{r}_1 \cdot \vec{r}_2)dt = \frac{4}{3}$

[2069 New]

⇒ Solution,

Given,

$\vec{r}_1 = 2t^2\vec{i} + 3(t-1)\vec{j} + 4t^2\vec{k}$

$\vec{r}_2 = (t-1)\vec{i} + t^2\vec{j} + (t-2)\vec{k}$

Now,

$$\vec{r}_1 \cdot \vec{r}_2 = 2t^2(t-1) + 3(t-1)(t^2) + 4t^2(t-2)$$

$$= 2t^3 - 2t^2 + 3t^3 - 3t^2 + 4t^3 - 8t^2$$

$$= 9t^3 - 13t^2$$

Now,

$$\int_0^2 (\vec{r}_1 \cdot \vec{r}_2)dt = \int_0^2 (9t^3 - 13t^2)dt$$

$$= 9\left[\frac{t^4}{4}\right]_0^2 - 13\left[\frac{t^3}{3}\right]_0^2$$

$$= 9\left[\frac{\overset{16}{\cancel{256}}}{4}\right] - 13\left[\frac{8}{3}\right]$$

$$= 36 - \frac{104}{3}$$

$$= \frac{4}{3} \text{ proved}$$

17. Evaluate : $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)dt$ where

$\vec{r} = 2t^2\vec{i} - 4t\vec{j} - 3t^2\vec{k}$ [2073]

⇒ Solution,

Given vector is

$$\vec{r} = 2t^2\vec{i} - 4t\vec{j} - 3t^2\vec{k}$$

Diff. both sides w.r.to t, we get

$$\frac{d\vec{r}}{dt} = (4t)\vec{i} - 4\vec{j} - 6t\vec{k}$$

and, $\frac{d^2\vec{r}}{dt^2} = 4\vec{i} - 0\vec{j} - 6\vec{k} = 4\vec{i} - 6\vec{k}$

Now,

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^2 & -4t & 3t^2 \\ 4 & 0 & -6 \end{vmatrix}$$

$$= (+24t)\vec{i} + (12t^2 + 12t^2)\vec{j} + 16t\vec{k}$$

$$= 24t\vec{i} + 16t\vec{k}$$

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$$\therefore \int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \int_1^2 (24t\vec{i} + 16t\vec{k})\, dt$$

$$= \left[12t^2\vec{i} + 8t^2\vec{k}\right]_1^2$$

$$= \cancel{8}\ 36\vec{i} + 24\vec{k}$$

$$= 12(3\vec{i} + 2\vec{k})$$

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