Gradient, Divergence and curl

1. Define gradient of a scalar function and curl of a vector function. If $f = x^3 + y^3 + z^3 - 3xyz$; find curl (grad f). 20

⇒ Solution,

Gradient of a scalar function

If $\phi(x,y,z)$ be a scalar point function defined and differentiable in every point of the region in the space, then the gradient of ϕ denoted by ϕ or $\nabla\phi$ and defined as

$$\text{Grad}(\phi) = \nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \Sigma\vec{i}\phi_x$$

Curl of vector function:

Let $\vec{v}(x,y,z)$ be a vector point function defined and differentiable every point of region in the space then the curl of $\vec{v}$ denoted by curl $\vec{v} = \nabla \times \vec{v}$.

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

Where, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$$\text{curl}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k}$$

Next part

Let $f = x^3+y^3+z^3-3xyz$

~~Differentiall~~

partially Diff. with respect to x, y, z resp.

$$\frac{\partial f}{\partial x} = 3x^2 - 3yz$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3xz$$

$$\frac{\partial f}{\partial z} = 3z^2 + 3xy$$

Now,

$$\text{grad } f = \nabla f = \left(\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right)$$

$$= (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}$$

$\text{Curl}(\text{grad} f) =$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left((3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$\left[\frac{\partial}{\partial y}(3x^2-3xy)-\frac{\partial}{\partial z}(3y^2-3zx)\right]\vec{i}+\left[\frac{\partial}{\partial z}(3x^2-3yz)\right.$$

$$\left.-\frac{\partial}{\partial x}(3z^2-3xy)\right]\vec{j}+\left[\frac{\partial}{\partial x}(3y^2-3zx)-\frac{\partial}{\partial y}(3x^2-3yz)\right]\vec{k}$$

$$=(-3x+3x)\vec{i}+(-3y+3y)\vec{j}+(-3z+3z)\vec{k}$$

$$=0$$

curl(grad f) = 0

2. Define divergence of a vector function. Show that $\text{div}(\text{grad}\,\phi)=\sum\frac{\partial^2\phi}{\partial x^2}$ [2057]

⇒ Solution,

Divergence of a vector function

Let $\vec{v}=(x,y,z)$ be a vector point function defined and differentiable in every point of any region in the space. The divergence of $\vec{v}$ is denoted by $\vec{v}$ or $\nabla\cdot\vec{v}$ is defined as

$$\text{div}\,\vec{v}=\nabla\cdot\vec{v}$$

$$=\vec{i}\frac{\partial\vec{v}}{\partial x}+\vec{j}\frac{\partial\vec{v}}{\partial y}+\vec{k}\frac{\partial\vec{v}}{\partial z}$$

$$=\sum\vec{i}\,\vec{v}_x$$

If $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ then

$$\therefore \text{div } \vec{v} = \left(\vec{i}\frac{\partial \vec{v}}{\partial x} + \vec{j}\frac{\partial \vec{v}}{\partial y} + \vec{k}\frac{\partial \vec{v}}{\partial z}\right) \cdot \left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \sum \frac{\partial v_1}{\partial x}.$$

Next part

we have to show that

$$\text{div}(\text{grad } \phi) = \sum \frac{\partial^2 \phi}{\partial x^2}$$

$$\text{L.H.S} = \text{div}(\text{grad } \phi)$$

$$= \nabla \cdot (\nabla \phi)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\text{R.H.S.} = \sum \frac{\partial^2 \phi}{\partial x^2} \quad \text{Proved}$$

3. Define curl of a vector function. Give its physical meaning [2057, 2071]

$\Rightarrow$ Solution,

Curl of vector function

see Q.N. 1

Physical interpretation

Consider the linear velocity $\vec{v}$ as a vector function. Then, Page-5

Date ____
Page ____

we have

$\vec{v} = \vec{\omega} \times \vec{r}$

Let, $\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now,

$$\vec{v} = (\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}) \times (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\vec{i} + (\omega_3 x - \omega_1 z)\vec{j} + (\omega_1 y - \omega_2 x)\vec{k}$$

Now,

$\text{Curl } \vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\omega_2 z - \omega_3 y)\vec{i} + (\omega_3 x - \omega_1 z)\vec{j} + (\omega_1 y - \omega_2 x)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z)\right]\vec{i} + \left[\frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x)\right]\vec{j} + \left[\frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right]\vec{k}$$

$$= (\omega_1 + \omega_1)\vec{i} + (\omega_2 + \omega_2)\vec{j} + (\omega_3 + \omega_3)\vec{k}$$

$$= 2(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k})$$

$$= 2\vec{\omega}$$

4 If ϕ is scalar function and $\vec{a}$ is a vector function then show that $\text{curl}(\phi\vec{a}) = \phi\,\text{curl}\,\vec{a} + (\text{grad}\,\phi)\times\vec{a}$ [2054, 2064, 2075]

$\Rightarrow$ Solution,

$$\text{L.H.S.} = \text{Curl}(\phi\vec{a}) = \nabla\times(\phi\vec{a})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{a})$$

$$= \vec{i}\times\frac{\partial}{\partial x}(\phi\vec{a}) + \vec{j}\times\frac{\partial}{\partial y}(\phi\vec{a}) + \vec{k}\times\frac{\partial}{\partial z}(\phi\vec{a})$$

$$= \vec{i}\times\left[\phi\frac{\partial\vec{a}}{\partial x} + \vec{a}\frac{\partial\phi}{\partial x}\right] + \vec{j}\times\left[\phi\frac{\partial\vec{a}}{\partial y} + \vec{a}\frac{\partial\phi}{\partial y}\right] + \vec{k}\times\left[\phi\frac{\partial\vec{a}}{\partial z} + \vec{a}\frac{\partial\phi}{\partial z}\right]$$

$$= \phi\left[\vec{i}\times\frac{\partial\vec{a}}{\partial x} + \vec{j}\times\frac{\partial\vec{a}}{\partial y} + \vec{k}\times\frac{\partial\vec{a}}{\partial z}\right] + \left[\vec{i}\frac{\partial\phi}{\partial x}\times\vec{a} + \vec{j}\frac{\partial\phi}{\partial y}\times\vec{a} + \vec{k}\frac{\partial\phi}{\partial z}\times\vec{a}\right]$$

$$= \phi\left[\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{a}\right] + \left[\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi\times\vec{a}\right]$$

$$= \phi(\nabla\times\vec{a}) + (\nabla\phi)\times\vec{a}.$$

$$= \phi\,\text{curl}\,\vec{a} + (\text{grad}\,\phi)\times\vec{a}$$

Hence, proved

5. If $\vec{f} = (x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}$, show that $\vec{f} \cdot \text{curl } \vec{f} = 0$. [BSc 2054]

$\Rightarrow$ Solution,

Let $\vec{f} = (x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}$,

Then,

$\text{Curl } f = \nabla \times \vec{f}$

Then,

$$\text{Curl } \vec{f} = \nabla \times \vec{f}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left[(x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right]\vec{i} + \left[\frac{\partial}{\partial z}(x+y+1) - \frac{\partial}{\partial x}(-x-y)\right]\vec{j} + \left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]\vec{k}$$

$$= (-1)\vec{i} + (+1)\vec{j} - (1)\vec{k}$$

$$= -\vec{i} + \vec{j} - \vec{k}$$

Now,

$$\vec{f} \cdot \text{curl } \vec{f} = \left[(x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}\right] \cdot \left[-\vec{i} + \vec{j} - \vec{k}\right]$$

$$= -x-y-1+1+x+y$$

$$= 0 \text{ Proved}$$

6. If $f = x^3+y^3+z^3-3xyz$, find div (grad f).
[2055]

$\Rightarrow$ Solution,

Let $f = x^3+y^3+z^3-3xyz$

Diff. partially (1) with respect to x, y, z resp.

$\frac{\partial f}{\partial x} = 3x^2-3yz$, $\frac{\partial f}{\partial y} = 3y^2-3zx$

$\frac{\partial f}{\partial z} = 3z^2-3xy$

Now,

grad $f = \nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$

Div (grad f) $= \nabla \cdot (\nabla f)$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(3x^2-3yz)\vec{i} + (3y^2-3zx)\vec{j} + (3z^2-3xy)\vec{k}\right]$

$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$= 6x+6y+6z$

$= 6(x+y+z).$

7. Define gradient of scalar point function. Prove that $div(\phi\vec{a}) = \phi\, div\,\vec{a} + \vec{a}\cdot(grad\,\phi)$ [2055 2070]

⇒ Solution,

L.H.S. $= div(\phi\vec{a})$

$= \nabla\cdot(\phi\vec{a})$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\phi\vec{a})$

$= \vec{i}\frac{\partial}{\partial x}(\phi\vec{a}) + \vec{j}\frac{\partial}{\partial y}(\phi\vec{a}) + \vec{k}\frac{\partial}{\partial z}(\phi\vec{a})$

$= \vec{i}\left[\phi\frac{\partial\vec{a}}{\partial x} + \vec{a}\frac{\partial\phi}{\partial x}\right] + \vec{j}\left[\phi\frac{\partial\vec{a}}{\partial y} + \vec{a}\frac{\partial\phi}{\partial y}\right] + \vec{k}\left[\phi\frac{\partial\vec{a}}{\partial z} + \vec{a}\frac{\partial\phi}{\partial z}\right]$

$= \phi\left[\vec{i}\frac{\partial\vec{a}}{\partial x} + \vec{j}\frac{\partial\vec{a}}{\partial y} + \vec{k}\frac{\partial\vec{a}}{\partial z}\right] + \vec{a}\cdot\left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right]$

$= \phi\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{a} + \vec{a}\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

$= \phi(\nabla\vec{a}) + \vec{a}\cdot(\nabla\phi)$

$= \phi(div.\vec{a}) + \vec{a}\cdot(grad\,\phi)$

Proved

8. If $\vec{f} = (x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}$ then find div $\vec{f}$ and $\vec{f} \times$ curl $\vec{f}$.

$\Rightarrow$ Solution,

$\vec{f} = (x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}$

Now,

$$\text{div } \vec{f} = \nabla \cdot \vec{f}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left((x+y+z)\vec{i} + \vec{j} + (-x-y)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x+y+z) + \frac{\partial}{\partial y}(1) + \frac{\partial}{\partial z}(-x-y)$$

$$\therefore \text{div } \vec{f} = 1.$$

Also,

for $\vec{f} \times \text{curl } \vec{f} = \vec{f} \times (\nabla \times \vec{f})$

Here,

$$\text{curl}(\vec{f}) = \nabla \times \vec{f}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= -\vec{i} + \vec{j} - \vec{k}$$

$$\therefore \vec{f} \times \text{curl}\vec{f} = \left[(x+y+1)\vec{i} + \vec{j} + (-x-y)\vec{k}\right] \times \left[-\vec{i} + \vec{j} - \vec{k}\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x+y+1 & 1 & -x-y \\ -1 & 1 & -1 \end{vmatrix}$$

$$= (x+y-1)\vec{i} + (2x+2y+1)\vec{j} + (x+y+2)\vec{k}$$

Date ____
Page ____

9 If $\vec{V}$ be a vector point function, prove that div (curl $\vec{V}$) = 0 [2054, 2056]

⇒ Solution,

Let $\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$

Now,

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(V_1\vec{i} + V_2\vec{j} + V_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \vec{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \vec{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Also,

$$\text{div}\,(\text{curl}\,\vec{V}) = \nabla \cdot (\nabla \times \vec{V})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

= 0 Proved

10. If $\vec{a}$ and $\vec{b}$ are vector point functions show that $\text{div}(\vec{a}\times\vec{b}) = \vec{b}\cdot\text{curl}(\vec{a}) - \vec{a}\cdot(\text{curl}\,\vec{b})$

[2058, 2075, 2060, 2061, 2074]

⇒ Solution,

$$\text{L.H.S.} = \text{div}(\vec{a}\times\vec{b})$$

$$= \nabla\cdot(\vec{a}\times\vec{b})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{a}\times\vec{b})$$

$$= \vec{i}\,\frac{\partial}{\partial x}(\vec{a}\times\vec{b}) + \vec{j}\,\frac{\partial}{\partial y}(\vec{a}\times\vec{b}) + \vec{k}\,\frac{\partial}{\partial z}(\vec{a}\times\vec{b})$$

$$= \vec{i}\cdot\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x} + \frac{\partial\vec{a}}{\partial x}\times\vec{b}\right) + \vec{j}\left(\vec{a}\times\frac{\partial\vec{b}}{\partial y} + \frac{\partial\vec{a}}{\partial y}\times\vec{b}\right) + \vec{k}\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x} + \frac{\partial\vec{a}}{\partial z}\times\vec{b}\right)$$

$$= \left(\vec{i}\cdot\vec{a}\times\frac{\partial\vec{b}}{\partial x} + \vec{j}\cdot\vec{a}\times\frac{\partial\vec{b}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{b}}{\partial z}\right) + \left(\vec{i}\cdot\frac{\partial\vec{a}}{\partial x}\times\vec{b} + \vec{j}\,\frac{\partial\vec{a}}{\partial y}\times\vec{b} + \vec{k}\cdot\frac{\partial\vec{a}}{\partial z}\times\vec{b}\right)$$

$$= -\left(\vec{i}\cdot\frac{\partial\vec{b}}{\partial x}\times\vec{a} + \vec{j}\cdot\frac{\partial\vec{b}}{\partial y}\times\vec{a} + \vec{k}\cdot\frac{\partial\vec{b}}{\partial z}\times\vec{a}\right) + \left(\vec{i}\,\frac{\partial\vec{a}}{\partial x}\times\vec{b} + \vec{j}\,\frac{\partial\vec{a}}{\partial y}\times\vec{b} + \vec{k}\,\frac{\partial\vec{a}}{\partial z}\times\vec{b}\right)$$

$$= -\left(\vec{i}\times\frac{\partial\vec{b}}{\partial x}\cdot\vec{a} + \vec{j}\times\frac{\partial\vec{b}}{\partial y}\cdot\vec{b} + \vec{k}\times\frac{\partial\vec{b}}{\partial z}\cdot\vec{a}\right) + \left(\vec{i}\times\frac{\partial\vec{a}}{\partial x}\cdot\vec{b} + \vec{j}\times\frac{\partial\vec{a}}{\partial y}\cdot\vec{b} + \vec{k}\times\frac{\partial\vec{a}}{\partial z}\cdot\vec{b}\right)$$

$$= -\left[\left\{\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{b}\right\}\cdot\vec{a}\right]$$

$$+\left[\left\{\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{a}\right\}\cdot\vec{b}\right]$$

$$= -(\nabla\times\vec{b})\cdot\vec{a} + (\nabla\times\vec{a})\cdot\vec{b}$$

$$= \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$$

$$= \vec{b}\cdot(\text{curl}\,\vec{a}) - \vec{a}\cdot(\text{curl}\,\vec{b})$$ R.H.S proved

11. If $\vec{v} = e^{xyz}(\vec{i}+\vec{j}+\vec{k})$ then find the curl $\vec{v}$.
[2071, 2071 old]

⇒ Solution,

$\vec{v} = e^{xyz}(\vec{i}+\vec{j}+\vec{k})$

$$\text{curl}\,\vec{v} = \nabla\times\vec{v}$$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\left[e^{xyz}(\vec{i}+\vec{j}+\vec{k})\right]$$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\left(e^{xyz}\vec{i}+e^{xyz}\vec{j}+e^{xyz}\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}e^{xyz} - \frac{\partial}{\partial z}e^{xyz}\right] + \vec{j}\left[\frac{\partial}{\partial z}e^{xyz} - \frac{\partial}{\partial x}e^{xyz}\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}e^{xyz} - \frac{\partial}{\partial y}e^{xyz}\right]$$

$$= \left(xze^{xyz} - xye^{xyz}\right)\vec{i} + \left(xye^{xyz} - yze^{xyz}\right)\vec{j} + \left(yze^{xyz} - zxe^{xyz}\right)\vec{k}$$

$$= xe^{xyz}(z-y)\vec{i} + ye^{xyz}(x-z)\vec{j} + ze^{xyz}(y-x)\vec{k}$$

$$= e^{xyz}\left[x(z-y)\vec{i} + y(x-z)\vec{j} + z(y-x)\vec{k}\right].$$

12 If $\phi = \log(x^2+y^2+z^2)$, find curl(grad ϕ). [2065, 2070(old), 2072]

$\Rightarrow$ Solution,

$\phi = \log(x^2+y^2+z^2)$

Now,

$$\text{grad}\,\phi = \nabla\cdot\phi$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\log(x^2+y^2+z^2)\right)$$

$$= \left(\frac{1}{x^2+y^2+z^2}\cdot 2x\right)\vec{i} + \left(\frac{1}{x^2+y^2+z^2}\cdot 2y\right)\vec{j} + \left(\frac{1}{x^2+y^2+z^2}\cdot 2z\right)\vec{j}$$

$$= \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{x^2+y^2+z^2}$$

Also,

$$\text{curl}(\text{grad}\,\phi) = \nabla\times\nabla\cdot\phi$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(\frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{vmatrix}$$

Date ______
Page ______

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+z^2}\right)\right] +$$

$$\vec{j}\left[\frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial x}\left(\frac{2z}{x^2+y^2+z^2}\right)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2+z^2}\right)\right]$$

$$= \vec{i}\left[-\frac{4yz}{(x^2+y^2+z^2)^2} + \frac{4yz}{(x^2+y^2+z^2)^2}\right] + \vec{j}\left[-\frac{4zx}{(x^2+y^2+z^2)^2} +\right.$$

$$\left.\frac{4zx}{(x^2+y^2+z^2)^2}\right] + \vec{k}\left[-\frac{4xy}{(x^2+y^2+z^2)^2} + \frac{4xy}{(x^2+y^2+z^2)^2}\right]$$

$$= 0.$$

13 Prove that $\mathrm{Div}(\phi\vec{a}) = \phi(\mathrm{div}\,\vec{a}) + \vec{a}\cdot(\mathrm{grad}\,\phi)$; where ϕ is a scalar function of x, y, z [2065]

$\Rightarrow$ Solution,

$$\text{L.H.S.} = \mathrm{Div}(\phi\vec{a})$$

$$= \nabla\cdot(\phi\vec{a})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\phi\vec{a})$$

$$= \vec{i}\left[\phi\frac{\partial\vec{a}}{\partial x} + \vec{a}\frac{\partial\phi}{\partial x}\right] + \vec{j}\left[\phi\frac{\partial\vec{a}}{\partial y} + \vec{a}\frac{\partial\phi}{\partial x}\right]$$

$$+ \vec{k}\left[\phi\frac{\partial\vec{a}}{\partial z} + \vec{a}\frac{\partial\phi}{\partial z}\right]$$

$$= \phi\left[\vec{i}\frac{\partial\vec{a}}{\partial x}+\vec{j}\frac{\partial\vec{a}}{\partial y}+\vec{k}\frac{\partial\vec{a}}{\partial z}\right]+\vec{a}\cdot\left[\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right]$$

$$= \phi\left[\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right]\cdot\vec{a} + \vec{a}\cdot\left[\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right]\phi$$

$$\cancel{= (\phi\nabla)\vec{a} + \vec{a}(\nabla\phi)} = \phi(\nabla\cdot\vec{a}) + \vec{a}\cdot(\nabla\phi)$$

$$= \phi\,(\mathrm{Div}\,\vec{a}) + \vec{a}\cdot(\mathrm{grad}\,\phi)$$

Hence proved

14. If $f = ayz^2 + bzx^2 + cxy^2$ verify that $\mathrm{Curl}(\mathrm{grad}\, f) = 0$. [2054, 2070, 2073]

$\Rightarrow$ Solution

$f = ayz^2 + bzx^2 + cxy^2$

Now,

$\mathrm{grad}\, f = \nabla f$

$$= \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$= \vec{i}(2bzx + cy^2) + \vec{j}(2cxy + az^2) + \vec{k}(2ayz + bx^2)$$

Now,

$\mathrm{Curl}(\mathrm{grad}\, f) = \nabla \times \nabla f$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2bzx + cy^2 & 2cxy + az^2 & 2ayz + bx^2 \end{vmatrix}$$

$$= \vec{i}(2az - 2az) + \vec{j}(2bx - 2bx) + \vec{k}(2cy - 2cy)$$

$= 0$ ~~proved~~ verified

15. If $\vec{f} = x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$, find div $\vec{f}$ and curl $\vec{f}$ at $(1,-1,1)$. [2062, 2069, 2071]

$\Rightarrow$ Solution,

$\vec{f} = x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$

Now,

div $\vec{f} = \nabla \cdot \vec{f}$

$= \vec{i}\frac{\partial \vec{f}}{\partial x} + \vec{j}\frac{\partial \vec{f}}{\partial y} + \vec{k}$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}\right)$

$= \frac{\partial}{\partial x}x^2z - \frac{\partial}{\partial y}2y^3z^2 + \frac{\partial}{\partial z}xy^2z$

$= 2xz - 6y^2z^2 + xy^2$

At a point $(1,-1,1)$

div $\vec{f} = 2\times1\times1 - 6\times(-1)^2\times1^2 + 1\times(-1)^2$

$= 2 - 6 + 1$

$= -3$

Also,

curl $\vec{f} = \nabla\times\vec{f}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & -2y^3z^2 & xy^2z \end{vmatrix}$

$= \vec{i}(2xyz + 4y^3z) + \vec{j}(x^2 - y^2z) + \vec{k}(-0-0)$

$= (2xyz + 4y^3z)\vec{i} + (x^2 - y^2z)\vec{j} +$

At $(1,-1,1)$

$= 2(1)(-1)(1) + 4(-1)^3(1) + 1^2 - (-1)^2(1)$

$= (-2-4)\vec{i} + 0\vec{j}$

$= -6\vec{i}$.

16 Find the angle between the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at point $(2,-1,2)$. [2066]

⇒ Solution,

Let $\phi = x^2+y^2+z^2=9$, $\psi = x^2+y^2-z-3$

Diff. partially w.r.to x, y, z respectively.

$\phi_x = 2x$, $\phi_y = 2y$, $\phi_z = 2z$

$\psi_x = 2x$, $\psi_y = 2y$, $\psi_z = -1$

Grad $\phi = \phi_x\vec{i} + \phi_y\vec{j} + \phi_z\vec{k}$

$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$= 2(x\vec{i} + y\vec{j} + z\vec{k})$

Grad ϕ at $(2,-1,2)$

$= 2(2\vec{i} - 1\vec{j} + 2\vec{k})$

$= 4\vec{i} - 2\vec{j} + 4\vec{k}$

Thus, unit vector normal to the surface $\phi = \hat{n}_1$

$$= \frac{4\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{4^2+(-2)^2+4^2}}$$

$$= \frac{2(2\vec{i} - \vec{j} + 2\vec{k})}{\sqrt{36}}$$

$$\therefore \hat{n}_1 = \frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}$$

Also,

$$\text{Grad}\,\psi = \psi_x\vec{i} + \psi_y\vec{j} + \psi_z\vec{k}$$

$$= 2x\vec{i} + 2y\vec{j} - 1\vec{k}$$

Grad ψ at $(2, -1, 2)$

$$= (2\times 2)\vec{i} - (2\times 1)\vec{j} - (1\times 2)\vec{k}$$

$$= 4\vec{i} - 2\vec{j} - 1\vec{k}$$

So, unit vector normal to surface, $\psi = \hat{n}_2$

$$\hat{n}_2 = \frac{4\vec{i} - 2\vec{j} - 1\vec{k}}{\sqrt{16+4+1}} = \frac{4\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{21}}$$

we know,

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$$

$$= \left(\frac{2\vec{i} - \vec{j} + 2\vec{k}}{3}\right) \cdot \left(\frac{4\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{21}}\right)$$

$$= \frac{1}{3\sqrt{21}}\left(8 + 2 - 2\right)$$

$$= \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

17. If $\vec{a}$ is a constant vector then prove that
(I) $\vec{a}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{\vec{a}\cdot\vec{r}}{r^3}$

(II) $grad(\vec{r}\cdot\vec{a}) = \vec{a}$ [2067, 2071 new]

⇒ Solution,
Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $r^2 = x^2+y^2+z^2$
Diff. partially with respect x we get,
$$\frac{\partial r^2}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

or, $2r\frac{\partial r}{\partial x} = 2x$

or, $\frac{\partial r}{\partial x} = \frac{x}{r}$

Now,
$$\nabla\left(\frac{1}{r}\right) = \sum\vec{i}\frac{\partial}{\partial x}(r^{-1})$$
$$= \sum\vec{i}\frac{\partial r^{-1}}{\partial r}\cdot\frac{\partial r}{\partial x}$$
$$= \sum\vec{i}(-1)\cdot r^{-2}\cdot\frac{x}{r}$$
$$= \sum -\vec{i}\frac{x}{r^3}$$
$$= -\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r^3} = -\frac{\vec{r}}{r^3}$$

Thus,

$$\vec{a}\cdot\nabla\left(\frac{1}{r}\right)=\vec{a}\cdot\left(-\frac{\vec{r}}{r^3}\right)$$

$$=-\frac{\vec{a}\cdot\vec{r}}{r^3}$$

(II) Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

Then, $\frac{\partial\vec{r}}{\partial x}=\vec{i}$

Now,

$$\text{grad}(\vec{r}\cdot\vec{a})=\nabla(\vec{r}\cdot\vec{a})$$

$$=\sum\vec{i}\,\frac{\partial}{\partial x}(\vec{r}\cdot\vec{a})$$

$$=\sum\vec{i}\left[\frac{\partial\vec{r}}{\partial x}\cdot\vec{a}+\vec{r}\cdot\frac{\partial\vec{a}}{\partial x}\right]=\sum\vec{i}\left[\frac{\partial\vec{r}}{\partial x}\cdot\vec{a}\right]$$

$$=\sum\vec{i}\,(\vec{i}\cdot\vec{a})$$

$$=\sum a_1\vec{i}=\vec{a}$$

Hence, proved.

18 Show that : $\operatorname{div}(\hat{r}) = \frac{2}{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
[2068 Old]

⇒ Solution,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

or, $r^2 = x^2 + y^2 + z^2$

or, $r = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \operatorname{div}(\hat{r}) = \nabla \cdot \left(\frac{\vec{r}}{r}\right)$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x}\left(\frac{\vec{r}}{r}\right)$$

$$= \sum \vec{i} \cdot \left[r\frac{\partial \vec{r}}{\partial x} - \vec{r}\frac{\partial r}{\partial x}\right]\frac{1}{r^2}$$

$$= \sum \vec{i} \cdot \left[r(\vec{i}) - \vec{r}\left(\frac{x}{r}\right)\right]\frac{1}{r^2}$$

$$= \frac{\sum \vec{i} \cdot \vec{i}}{r} - \frac{\sum (\vec{i} \cdot \vec{r})x}{r^3}$$

$$= \frac{\sum 1}{r} - \frac{\sum x^2}{r^3}$$

$$= \frac{3}{r} - \frac{r^2}{r^3}$$

$$= \frac{2}{r}$$

Hence, Proved

Date ____ Page ____

19 Find the unit vector normal to the surface $z = x^2 + y^2$ at the point $(-1, -2, 5)$ [2072]

⇒ solution,

The given surface be $\phi = z - x^2 - y^2$

$$\therefore \text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(z - x^2 - y^2)$$

$$= \vec{i}\,(-2x) + \vec{j}\,(-2y) + \vec{k}.1$$

$$= 2x\vec{i} - 2y\vec{j} + \vec{k}$$

At grad ϕ at $(-1, -2, 5)$

$$= 2\vec{i} + 4\vec{j} + \vec{k}$$

Hence, unit vector to the surface at the point $(-1, -2, 5)$ is

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2\vec{i} + 4\vec{j} + \vec{k}}{|2\vec{i} + 4\vec{j} + \vec{k}|}$$

$$= \frac{2\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{4 + 16 + 1}}$$

$$= \frac{2\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{21}}$$

$$= \frac{2}{\sqrt{21}}\vec{i} + \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}.$$

20 For any space vector $\vec{v}$ prove that
$$\text{grad}(\text{div}\,\vec{v}) = \text{Curl}(\text{curl}\,\vec{v}) + \sum \frac{\partial^2 v}{\partial x^2} \quad [2072]$$

⇒ Solution,
Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ Then $\nabla = i\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$$\text{Curl}(\text{curl}\,\vec{v}) = \nabla \times (\nabla \times \vec{v})$$
$$= \nabla \times \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left[\vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \vec{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\right]$$

$$= \sum \left[\frac{\partial^2 v_2}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 v_3}{\partial z \partial x}\right]\vec{i}$$

Date ____
Page ____

$$= \sum \left[\left(\frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} \right) \vec{i} \quad \left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \vec{i} \right]$$

Now, adding $\frac{\partial^2 V_1}{\partial x^2} \vec{i}$, we get

$$= \sum \left[\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} \right] \right.$$

$$= \sum \left[\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial y \partial x} + \frac{\partial^2 V_3}{\partial z \partial x} \right) \vec{i} \quad \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2} \right) \vec{i} \right]$$

$$= \sum \frac{\partial}{\partial x} (\nabla \cdot \vec{V}) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \sum \vec{i} V_1$$

$$= \nabla (\nabla \cdot \vec{V}) - \left(\sum \vec{i} \frac{\partial^2}{\partial x^2} \right) \vec{V}$$

$$= \text{grad} (\text{div} \vec{V}) - \sum \frac{\partial^2 V}{\partial x^2}$$

Proved $\text{curl} (\text{curl} \vec{V}) = \text{grad} (\text{div} \vec{V}) - \sum \frac{\partial^2 \vec{V}}{\partial x^2}$

Hence, $\text{grad} (\text{div} \vec{V}) = \text{curl} (\text{curl} \vec{V}) + \sum \frac{\partial^2 \vec{V}}{\partial x^2}$

21 Define the term solenoidal and irrotational. If $\vec{a}$ and $\vec{b}$ are irrotational show that $\vec{a}\times\vec{b}$ is solenoidal [2072]

⇒ Solution,

Solenoidal : A vector function $\vec{v}$ is said to be solenoidal if its divergent is zero, i.e. $\nabla\cdot\vec{v}=0$.

Irrotational : A vector function $\vec{v}$ is said to be irrotational if its curl is zero, i.e. $\nabla\times\vec{v}=0$.

Given, $\vec{a},\vec{b}$ are irrotational

$\therefore \nabla\times\vec{a}=0$, and $\nabla\times\vec{b}=0$

Now,

$$\nabla\cdot(\vec{a}\times\vec{b}) = \sum\vec{i}\,\frac{\partial}{\partial x}\cdot(\vec{a}\times\vec{b})$$

$$= \sum\vec{i}\,\frac{\partial}{\partial x}(\vec{a}\times\vec{b})$$

$$= \sum\vec{i}\cdot\left[\vec{a}\times\frac{\partial\vec{b}}{\partial x}+\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right]$$

$$= -\sum\vec{i}\cdot\left(\frac{\partial\vec{b}}{\partial x}\times\vec{a}\right)+\sum\vec{i}\times\left(\frac{\partial\vec{a}}{\partial x}\cdot\vec{b}\right)$$

$$= \sum\vec{i}\times\left(\frac{\partial\vec{b}}{\partial x}\cdot\vec{a}\right)+\sum\vec{i}\times\left(\frac{\partial\vec{a}}{\partial x}\cdot\vec{b}\right)$$

$$\therefore \nabla\cdot(\vec{a}\times\vec{b}) = -(\nabla\times\vec{b})\cdot\vec{a}+(\nabla\times\vec{a})\cdot\vec{b}$$

$$= -0\cdot\vec{a}+0\cdot\vec{b}$$

$$= 0$$

Hence $(\vec{a}\times\vec{b})$ is solenoidal.

22 Show that (i) $(\vec{a} \cdot \nabla)\phi = \vec{a} \cdot (\nabla \phi)$

(ii) $(\vec{a} \cdot \nabla)\vec{r} = \vec{a}$

where ϕ is the function of x, y, z.

⇒ Solution, [2073]

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ be a vector.

(i) L.H.S

$$(\vec{a} \cdot \nabla)\phi = \left[(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\right]\phi$$

$$= \left(a_1\frac{\partial \phi}{\partial x} + a_2\frac{\partial \phi}{\partial y} + a_3\frac{\partial \phi}{\partial z}\right)$$

$$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \vec{a} \cdot \nabla\phi$$

= R.H.S proved

(II) L.H.S $= (\vec{a} \cdot \nabla)\vec{r}$

$$= \left[(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\right]\vec{r}$$

$$= a_1\frac{\partial \vec{r}}{\partial x} + a_2\frac{\partial \vec{r}}{\partial y} + a_3\frac{\partial \vec{r}}{\partial z}$$

$$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$$= \vec{a}$$

= R.H.S proved

2077/09/15