→ 4marks

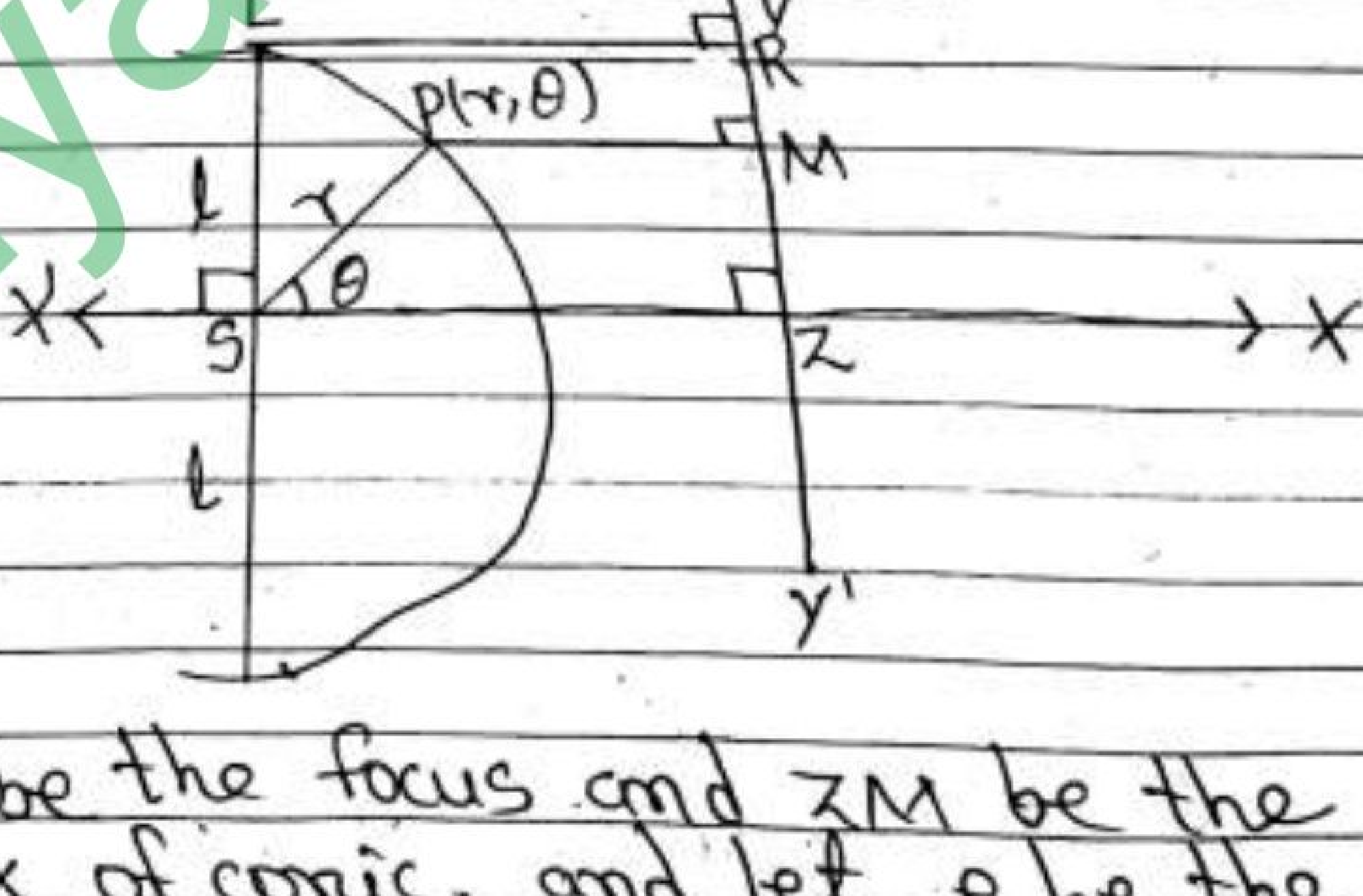
unit 3 Polar Equation of a Conic

Q.N.1. Define conic. Obtain the polar equation of a conic having eccentricity e & latus rectum 2l and focus being taken as pole. Also, find the equation of directrix. [2055, 2059, 2068old, 2056, 2058, 2071 old, 2071 New]

⇒ Solution

The locus of a point which moves in plane such that its distance from a fixed ~~straight~~ line point is a constant ratio to its perpendicular distance from a fixed straight line is called Conic section. The fixed point is called focus and the straight line is called

Polar equation of conic



Let S be the focus and ZM be the directrix of conic. and let e be the

eccentricity and l be the semi latus rectum.
Draw $SX \perp ZM$ and take SX as the intial line. Let $P(r, \theta)$ be any point on the conic reffered to S as pole.
Also, Draw $PM \perp ZM$ and $PN \perp SX$.
Suppose LSL' be the latus rectum.
Then, $LS = L'S$ = semi latus rectum $= l$
Now,
From the figure,

$$\frac{SP}{PM} = e$$

or, $SP = e PM$

or, $r = e NZ \quad [\because PM = NZ]$

or, $r = e(SZ - SN)$

$\therefore r = e(LR - SN) \longrightarrow (1)$

Also,

$$\frac{LS}{LR} = e$$

$$LR = \frac{LS}{e} = \frac{l}{e} \longrightarrow (2)$$

From rt. ΔPSN,

$$\cos\theta = \frac{SN}{SP} = \frac{SN}{r}$$

$\therefore SN = r\cos\theta \longrightarrow (3)$

Now, eqn. (1) becomes,

$r = e\left(\frac{l}{e} - r\cos\theta\right)$

or, $r = e\cdot\frac{l}{e} - re\cos\theta$

or, $l = r + re\cos\theta$

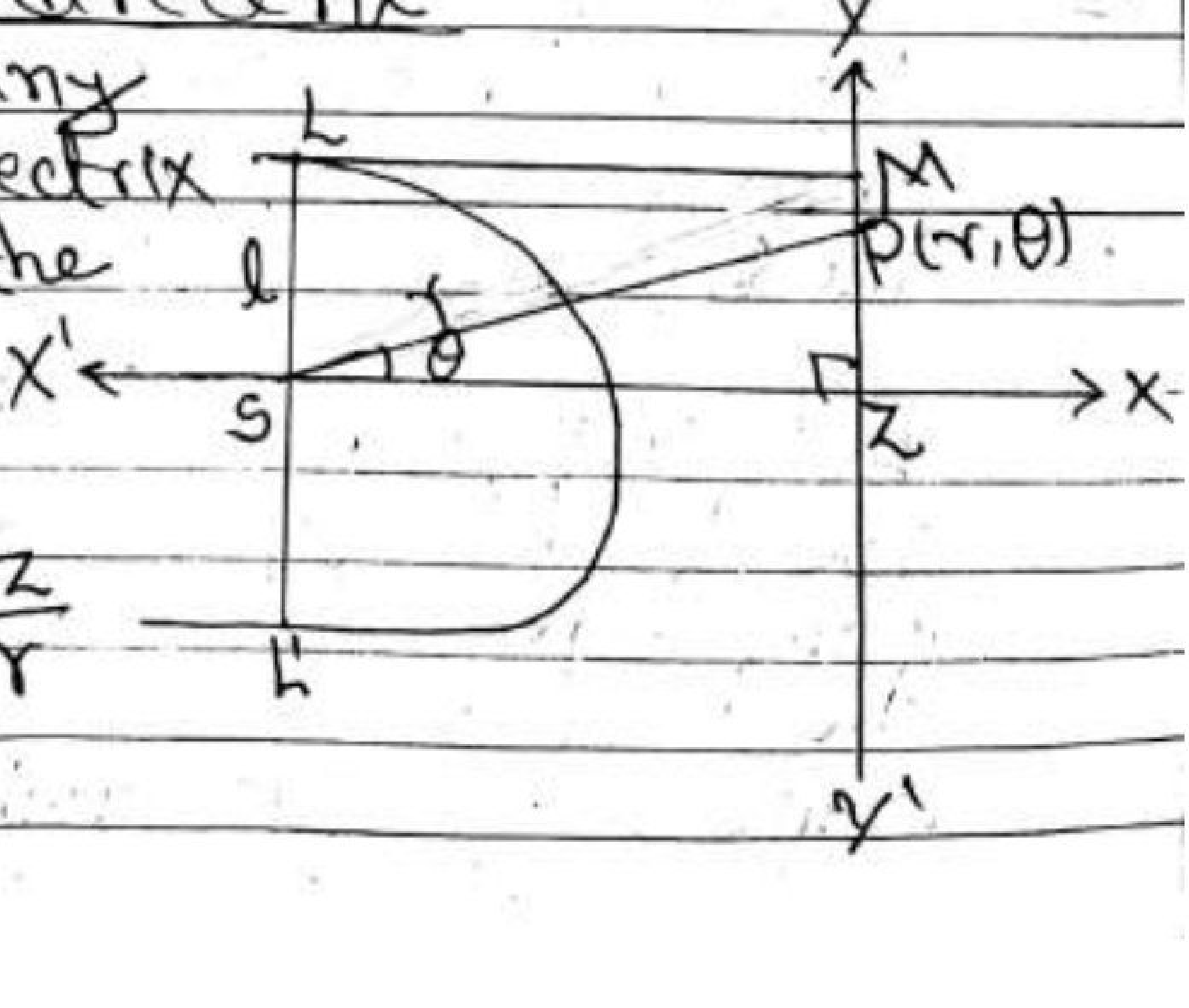
or, $l = r(1 + e\cos\theta)$

or, $\frac{l}{r} = 1 + e\cos\theta$

Which is the required eqn. of conic in polar form.

Equation of directrix

Let $P(r,\theta)$ be any point on the directrix corresponding to the focus S.



Then,

$\cos\theta = \frac{SZ}{SP} = \frac{SZ}{r}$

$SZ = r\cos\theta$

Also, by the definition of conic;

$SL = eLM$

or, $l = eSZ$

or, $l = er\cos\theta$

or, $\frac{l}{r} = e\cos\theta$

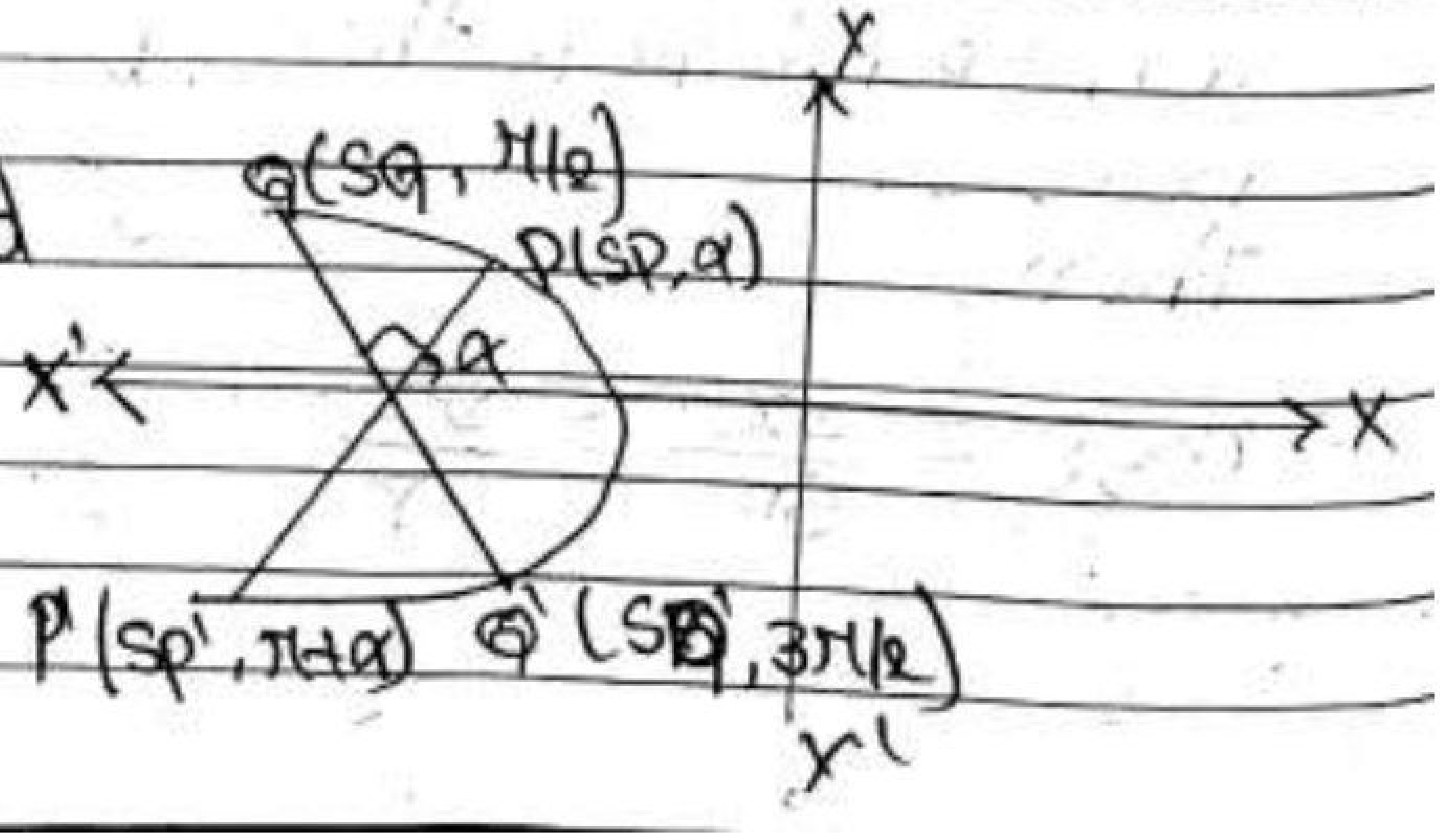
Which is the required equation of directrix.

2. What is focal chord of a conic? If PSP' and QSQ' are two perp. focal chords of a conic; then prove that $\frac{1}{PS \cdot SP'} + \frac{1}{QS \cdot SQ'}$ is a constant [2061]

⇒ Solution,

Any chord of the conic which passes through focus is called focal chord of the conic.

Let PSP' and QSQ' be two perp focal chords of a conic.



$$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$$

Let the vectorial angle of p is α. Then the vectorial angle of p' will be $\pi + \alpha$.

Therefore,

$$\frac{l}{SP} = 1 + e\cos\alpha$$

and

$$\frac{l}{SP'} = 1 + e\cos(\pi + \alpha)$$

i.e. $\frac{1}{SP} = \frac{1 + e\cos\alpha}{l}$

and

$$\frac{1}{SP'} = \frac{1 - e\cos\alpha}{l}$$

Again if the vectorial angle of Q is $\frac{\pi}{2} + \alpha$ then the vectorial angle of Q' will be $\left(\pi + \frac{\pi}{2} + \alpha\right) = \left(\frac{3\pi}{2} + \alpha\right)$

So, from eq^n (1)

$$\frac{l}{QS} = 1 + e\cos(\pi/2 + \alpha)$$

and

$$\frac{l}{Q'S} = 1 + e\cos\left(\frac{3\pi}{2} + \alpha\right)$$

or, $\frac{l}{SQ} = 1 - e\sin\alpha$ and $\frac{1}{SQ'} = 1 + e\sin\alpha$

or, $\frac{1}{SQ} = \frac{1 - e\sin\alpha}{l}$ and $\frac{1}{SQ'} = \frac{1 + e\sin\alpha}{l}$

Now,

$\frac{1}{PS} \cdot \frac{1}{SP'} + \frac{1}{QS} \cdot \frac{1}{SQ'}$

$= \left(\frac{1 + e\cos\alpha}{l}\right)\left(\frac{1 - e\cos\alpha}{l}\right) + \left(\frac{1 - e\sin\alpha}{l}\right)\left(\frac{1 + e\sin\alpha}{l}\right)$

$= \frac{1 - e^2\cos^2\alpha}{l^2} + \frac{1 - e^2\sin^2\alpha}{l^2}$

$= \frac{2 - e^2(\sin^2\alpha + \cos^2\alpha)}{l^2}$

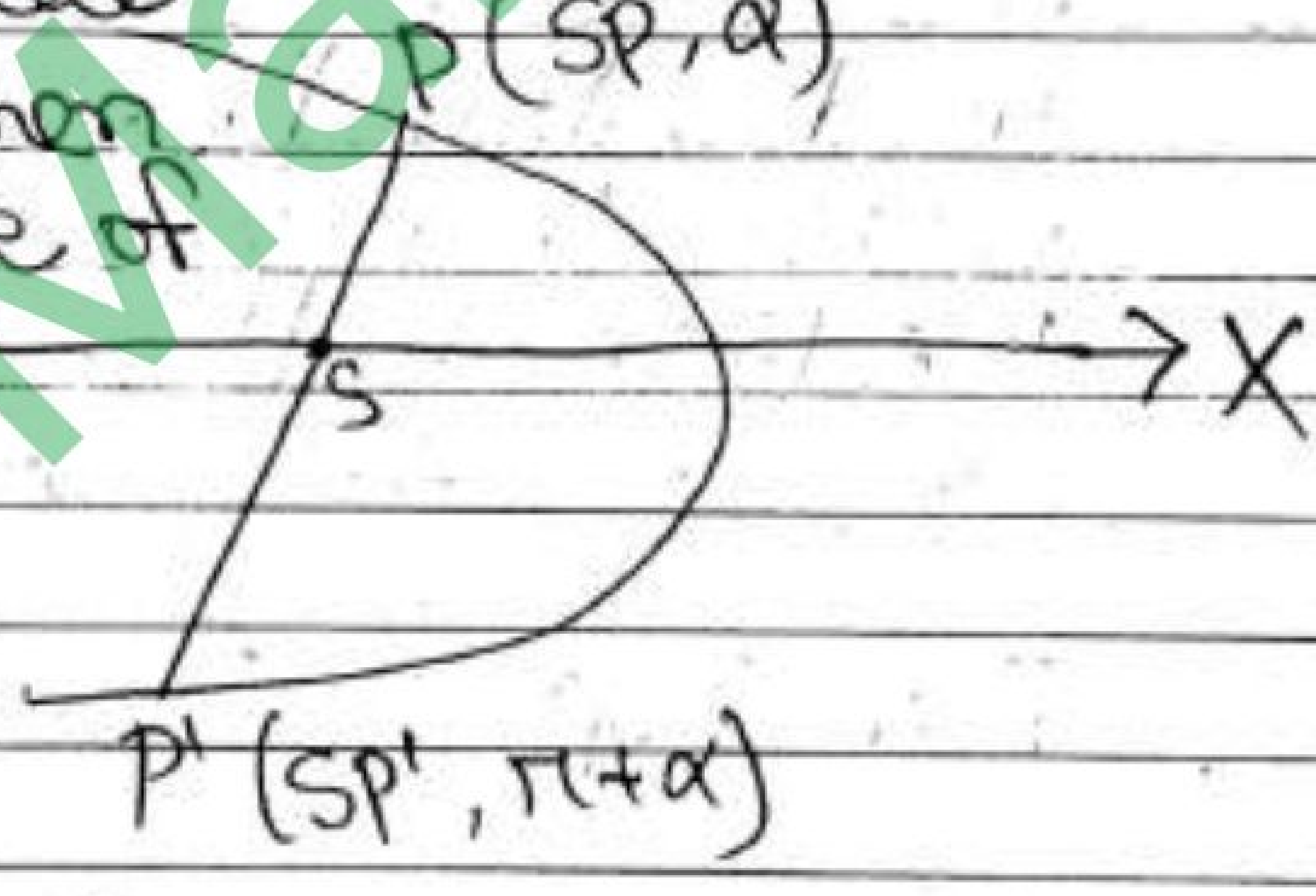
$= \frac{2 - e^2}{l^2}$, which is constant.

Hence proved

3 In any conic section, prove that the sum of the reciprocals of the segments of any focal chord is constant.
[2054, 2055, 2056, 2058, 2059, 2063, 2064, 2071, 2075]

4 Prove that in any conic section, the semi latus rectum is a harmonic mean between the segments of any focal chord.

⇒ Solution
Let, PSP' be a focal chord of a conic, then the vectorial angle of P' is $(\pi+\alpha)$.



so,
$$\frac{1}{SP} = \frac{1+e\cos\alpha}{l}$$

and
$$\frac{1}{SP'} = \frac{1+e\cos(\pi+\alpha)}{l} = \frac{1-e\cos\alpha}{l}$$

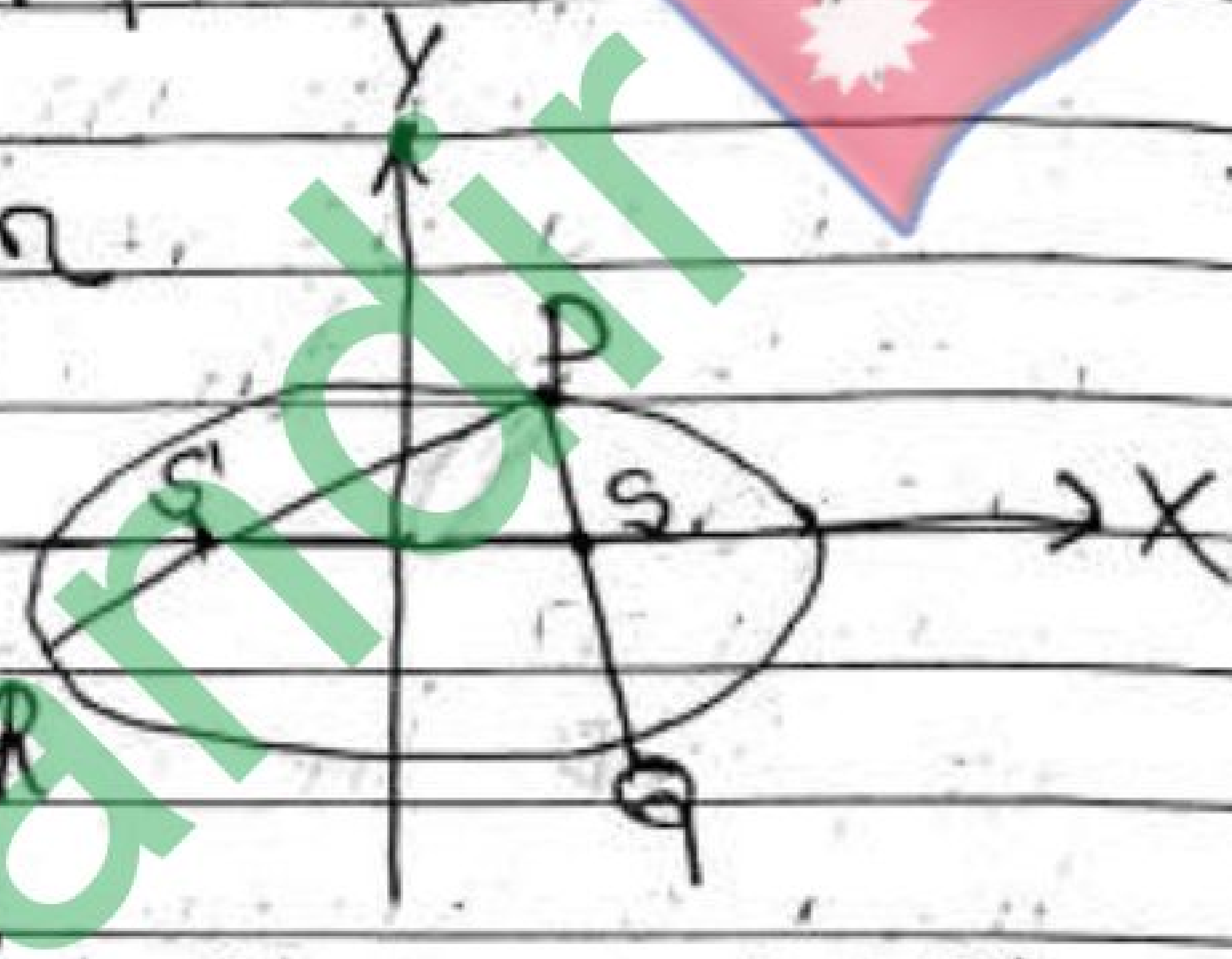
Now,
$$\frac{1}{SP} + \frac{1}{SP'} = \frac{1+e\cos\alpha+1-e\cos\alpha}{l}$$
$$= \frac{2}{l} \text{ which is constant}$$

Proved

4. If PSQ and PS'R be two chords of an ellipse through the foci S and S', show that $\frac{PS}{SQ} + \frac{PS'}{S'R}$ is independent of the position of P.

⇒ Solution,

Let p be a point on ellipse whose vectorial angle is α. PSQ and PS'R are two focal chords, then the vectorial angle of Q is $\pi+\alpha$



We know, the polar eqn of conic is

$$\frac{l}{r} = 1+e\cos\theta$$

For SP

$$\frac{l}{SP} = 1+e\cos\alpha$$

For SQ

$$\frac{l}{SQ} = 1+e\cos(\pi+\alpha) = 1-e\cos\alpha$$

Now,

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1+e\cos\alpha+1-e\cos\alpha}{l}$$

$$= \frac{2}{l}$$

Multyplying by SP.

$$1 + \frac{SP}{SQ} = \frac{2}{l} \cdot SP$$

or, $$\frac{SP}{SQ} = \frac{2}{l} SP - 1 \longrightarrow (1)$$

Similarly,

$$\frac{S'P}{RS'} = \frac{2}{l} S'P - 1 \longrightarrow (2)$$

Adding eq^n (1) and (2), we get

$$\frac{SP}{SQ} + \frac{S'P}{RS'} = \frac{2}{l}(SP + S'P) - 2$$

or, $$\frac{PS}{SQ} + \frac{PS'}{S'R} = \frac{2}{l} 2a - 2$$

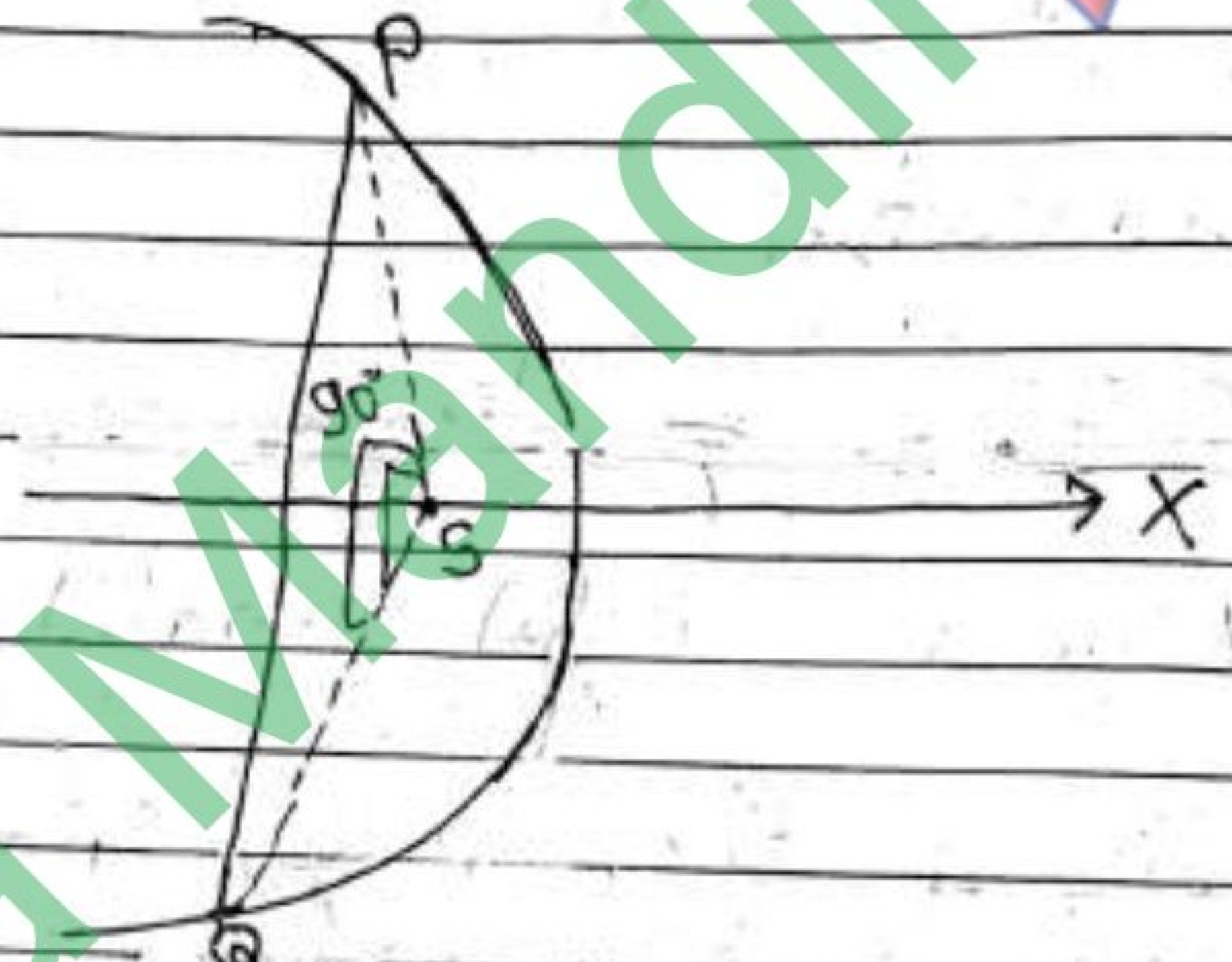
$[\because SP + S'P = 2a]$

or, $$\frac{PS}{SQ} + \frac{PS'}{S'R} = \frac{4a}{l} - 2$$

which is constant ie. it is independent of position of p.

5. If a chord PQ of a conic whose eccentricity is e and semi latus rectum is l subtends a right angle at the focus S, prove that
$$\left(\frac{1}{SP}-\frac{1}{l}\right)^2+\left(\frac{1}{SQ}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}$$

⇒ Solution,



Let the conic be $\frac{l}{r}=1+e\cos\theta$.
Let the vectorial angle of P is α.

$$\frac{l}{SP}=1+e\cos\alpha$$

or, $$\frac{1}{SP}=\frac{1+e\cos\alpha}{l}$$

Similarly, for Q
$$\frac{1}{SQ}=\frac{1+e\cos(\pi/2+\alpha)}{l}$$

$$\frac{1}{SQ} = \frac{1-e\sin\alpha}{l}$$

Now, taking L.H.S.

$$\left(\frac{1}{SP}-\frac{1}{l}\right)^2+\left(\frac{1}{SQ}-\frac{1}{l}\right)^2$$

$$=\left(\frac{1+e\cos\alpha-1}{l}\right)^2+\left(\frac{1-e\sin\alpha-1}{l}\right)^2$$

$$=\frac{e^2\cos^2\alpha}{l^2}+\frac{e^2\sin^2\alpha}{l^2}$$

$$=\frac{e^2}{l^2}\left(\sin^2\alpha+\cos^2\alpha\right)$$

$$=\frac{e^2}{l^2}$$ Proved

6. Find the equation of the chord of the conic $\frac{l}{r}=1+e\cos\theta$, joining the points whose vectorial angles are $\frac{\pi}{6}$ and $\frac{\pi}{2}$.

⇒ Solution,

Given conic is

$$\frac{l}{r}=1+e\cos\theta$$

Let $\theta_1 = \pi/6$ and $\theta_2 = \pi/2$ be the vectorial angles of points of ends of the chord. Then the eqn of chord is

$$\frac{l}{r} = \sec\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\theta - \frac{\theta_2 + \theta_1}{2}\right) + e\cos\theta$$

or, $\frac{l}{r} = \sec\left(\frac{\pi/2 - \pi/6}{2}\right)\cos\left(\theta - \frac{\pi/2 + \pi/6}{2}\right) + e\cos\theta$

or, $\frac{l}{r} = \sec\pi/6 \cdot \cos\left(\theta - \frac{\pi}{3}\right) + e\cos\theta$

or, $\frac{l}{r} = \frac{2}{\sqrt{3}}\left(\cos\theta \cdot \cos\frac{\pi}{3} + \sin\theta \cdot \sin\frac{\pi}{3}\right) + e\cos\theta$

or, $\frac{l}{r} = \frac{2}{\sqrt{3}}\left(\cos\theta \cdot \frac{1}{2} + \sin\theta \cdot \frac{\sqrt{3}}{2}\right) + e\cos\theta$

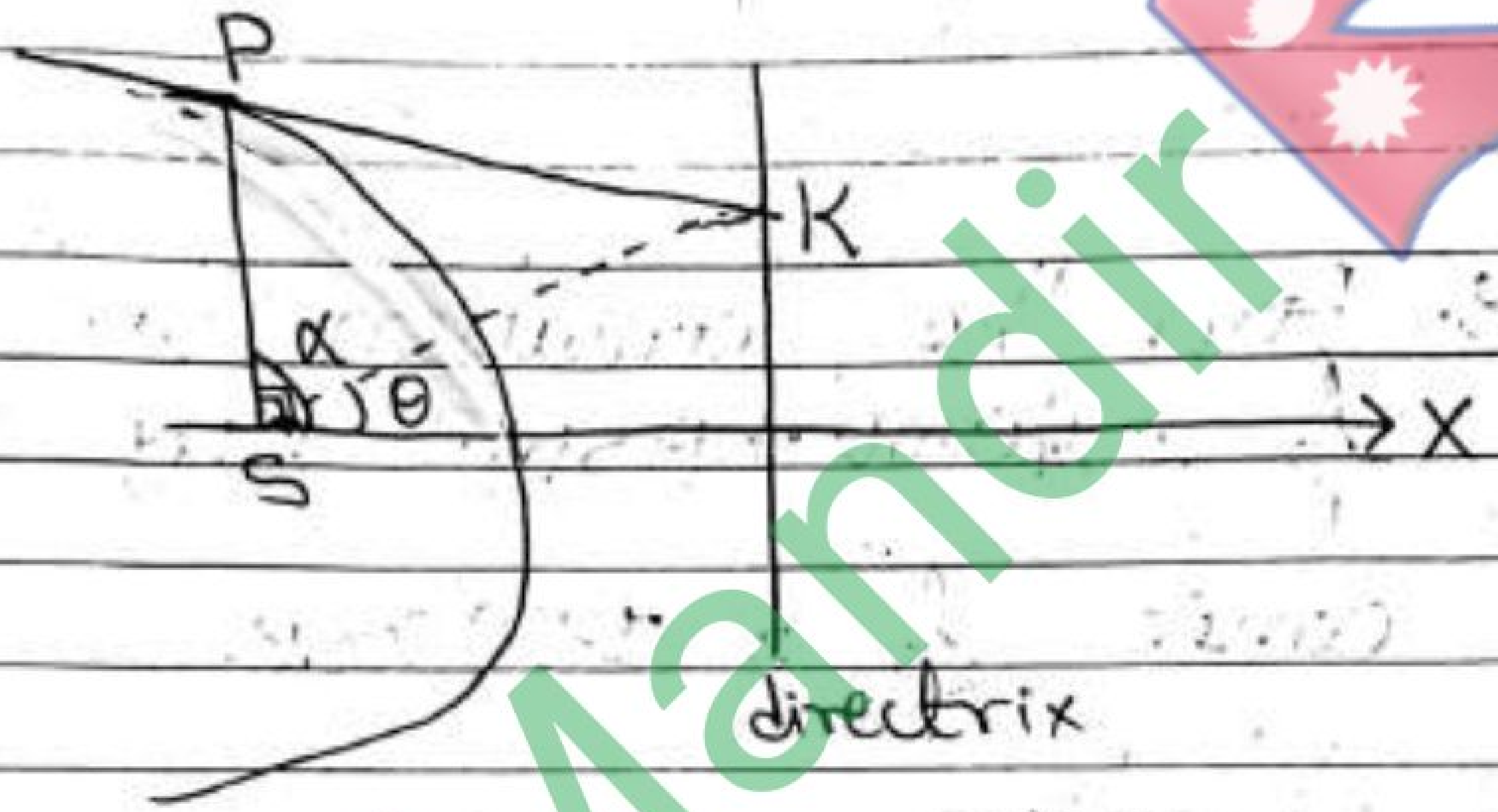
or, $\frac{l}{r} = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta + e\cos\theta$

or, $\frac{l}{r} = \cos\theta\left(e + \frac{1}{\sqrt{3}}\right)\cos\theta + \sin\theta$

which is required equation.

4. If the tangent at any point P of a conic meet the directrix in K, prove that KSP is a right angle.

⇒ solution,



Let the conic is $\frac{l}{r} = 1 + e\cos\theta$.

Let P be the point on conic and the tangent at P meets directrix at K. Let the vectorial angles of P and K be α and θ resp. then we have to show

$\alpha \sim \theta = 90°$

Here, the equation of tangent at P is

$\frac{l}{r} = \cos(\theta - \alpha) + e\cos\theta$

Again the equation of directrix is

$\frac{l}{r} = e\cos\theta$

Equating,

$\cos(\theta-\alpha)+e\cos\theta = e\cos\theta$

$\cos(\theta-\alpha) = 0$

$\theta-\alpha = 90^\circ$

$\therefore \alpha - \theta = 90^\circ.$

Hence $\angle KSP$ is a right angle

8. Find the condition that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may touch the conic $\frac{l}{r} = 1+e\cos\theta$.

$\Rightarrow$ Solution,

Given line is

$\frac{l}{r} = A\cos\theta + B\sin\theta \longrightarrow (1)$

and conic is

$\frac{l}{r} = 1+e\cos\theta \longrightarrow (2)$

Here, the equation of tangent to the conic at any point P is

$\frac{l}{r} = \cos(\theta-\alpha)+e\cos\theta$

$= \cos\theta\cdot\cos\alpha+\sin\theta\cdot\sin\alpha + e\cos\theta \longrightarrow (3)$

$= (\cos\alpha+e)\cos\theta + \sin\theta\cdot\sin\alpha$

If the line (1) is tangent to the conic

then eqn (1) and (3) must be identical

$\therefore A = e + \cos\alpha$ and $B = \sin\alpha$

or, $\cos\alpha = A - e$.

We have,

$\sin^2\alpha + \cos^2\alpha = 1$

$B^2 + (A-e)^2 = 1$

Which is required condition.

9. Find the equation of tangent whose vectorial angle is α for conic $\frac{l}{r} = 1 + e\cos\theta$. [TU 2072]

⇒ Solution,

Given equation of conic is

$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$

Let $P(r_1, \alpha)$ and $Q(r_2, \alpha')$ be any two point whose vectorial angle is α for the conic (1) resp.

So, the eqn. of chord PQ is :

$$\frac{l}{r} = \sec\left(\frac{\alpha' - \alpha}{2}\right)\cos\left(\theta - \frac{\alpha + \alpha'}{2}\right) + e\cos\theta \quad (2)$$

Since, the chord PQ will be tangent at P to the curve (1) if $Q \to P$ i.e $\alpha' \to \alpha$

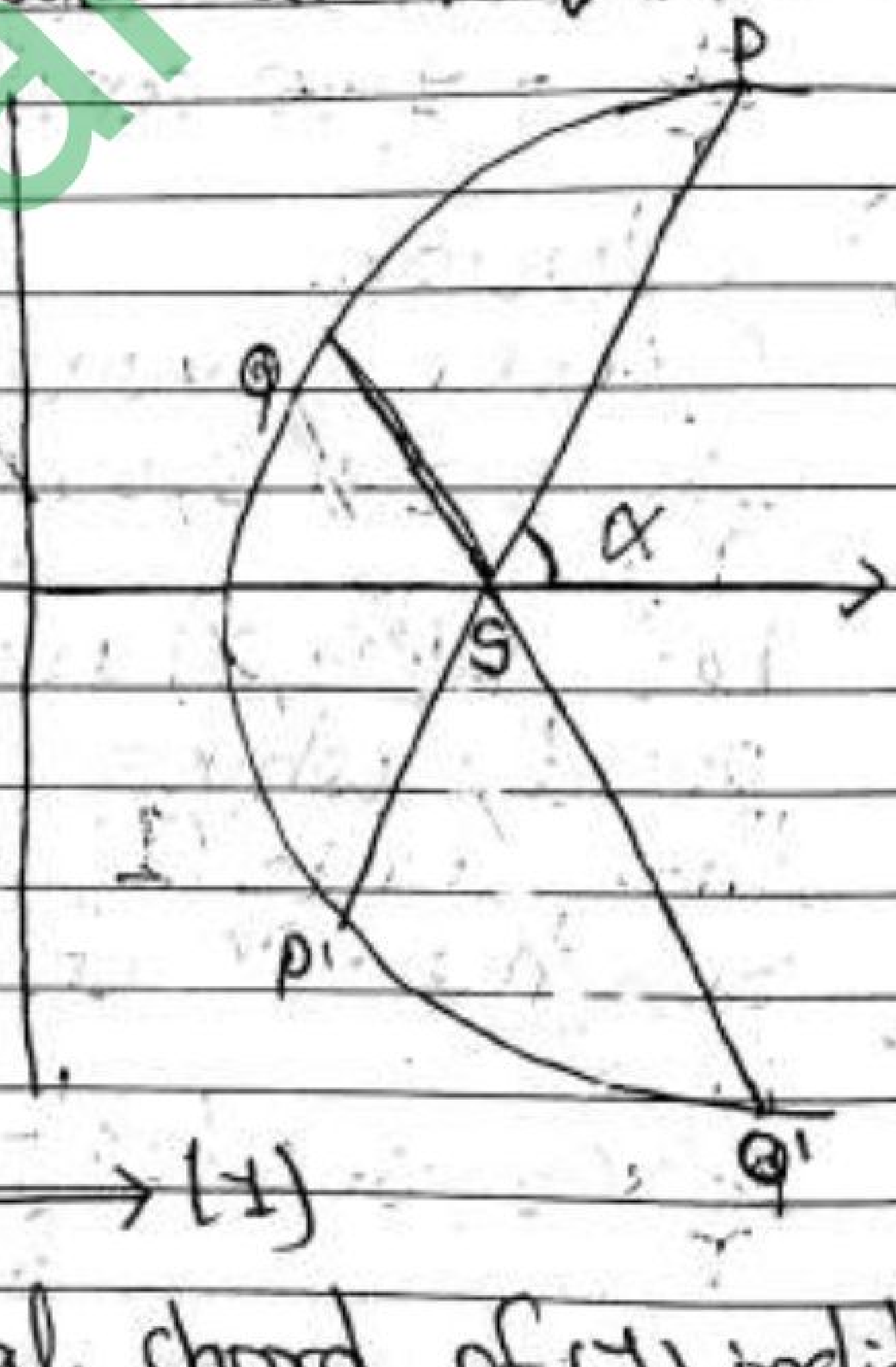
Hence the equation of tangent to the conic at P is

$$\frac{l}{r} = \cos(\theta-\alpha) + e\cos\theta$$

Which is required equation of tangent to the conic (1) whose vectorial angle is α.

10. Prove that perpendicular focal chords of rectangular hyperbola are equal.

$\Rightarrow$ Solution,



Since, the rectangular hyperbola, $e = \sqrt{2}$

The equation of conic is

$$\frac{l}{r} = 1 + e\cos\theta$$

becomes

$$\frac{l}{r} = 1 + \sqrt{2}\cos\theta$$

$$\therefore r = \frac{l}{1+\sqrt{2}\cos\theta} \longrightarrow (1)$$

Let PSP' be any focal chord of (1) included at an angle α. If the vectorial angle of P is α, then that of P' will be

$\pi+\alpha$.

$$\therefore SP = \frac{l}{1+\sqrt{2}\cos\alpha}$$

$$SP' = \frac{l}{1+\sqrt{2}\cos(\pi+\alpha)}$$

Now,

$$SP+SP' = \frac{l}{1+\sqrt{2}\cos\alpha} + \frac{l}{1+\sqrt{2}\cos(\pi+\alpha)}$$

$$= l\left[\frac{1}{1+\sqrt{2}\cos\alpha} + \frac{1}{1+\sqrt{2}\cos(\pi+\alpha)}\right]$$

$$= l\left[\frac{1}{1+\sqrt{2}\cos\alpha} + \frac{1}{1-\sqrt{2}\cos\alpha}\right]$$

$$= l\left[\frac{1-\sqrt{2}\cos\alpha+1+\sqrt{2}\cos\alpha}{(1+\sqrt{2}\cos\alpha)(1-\sqrt{2}\cos\alpha)}\right]$$

$$= \frac{2l}{1-2\cos^2\alpha}$$

$$= \frac{-2l}{2\cos^2\alpha-1}$$

$$\therefore PSP' = \frac{2l}{\cos 2\alpha} \longrightarrow (1)$$

Also,
Let QSQ' be other focal chord perp. to PSP'. If the vectorial angle of Q is $\left(\frac{\pi}{2}+\alpha\right)$, then that of Q' will be

$$\left(\pi+\frac{\pi}{2}+\alpha\right), \text{ i.e } \left(\frac{3\pi}{2}+\alpha\right)$$

So, $$QS = \frac{l}{1+\sqrt{2}\cos\left(\frac{\pi}{2}+\alpha\right)}$$

and $$SQ' = \frac{l}{1+\sqrt{2}\cos\left(\frac{3\pi}{2}+\alpha\right)}$$

Now,

$$QS + SQ' = \frac{l}{1+\sqrt{2}\cos\left(\frac{\pi}{2}+\alpha\right)} + \frac{l}{1+\sqrt{2}\cos\left(\frac{3\pi}{2}+\alpha\right)}$$

$$= l\left(\frac{1}{1-\sqrt{2}\sin\alpha} + \frac{1}{1+\sqrt{2}\sin\alpha}\right)$$

$$= l\left(\frac{2}{1-2\sin^2\alpha}\right)$$

$$\therefore QSQ' = \frac{2l}{\cos 2\alpha} \longrightarrow (2)$$

From eqn. (1) and (2), it is clear that perp. focal chord of rect. hyperbola are equal,

11. Prove that the equations $\frac{l}{r} = 1 + e\cos\theta$ and $\frac{l}{r} = e\cos\theta - 1$ represent the same conic. [2060, 2068, 2070, 2071].

⇒ Solution;

The given conic is

$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$

The points (r_1, θ_1) and $(-r_1, \pi + \theta_1)$ represent the same ~~conic~~ point in any conic.

Let $P(r_1, \theta_1)$ be any point on the conic (1). Then

$\frac{l}{r_1} = 1 + e\cos\theta_1 \longrightarrow (2)$

And given eqn. of other conic is

$\frac{l}{r} = e\cos\theta - 1 \longrightarrow (3)$

If the point $(-r_1, \pi + \theta_1)$ lies on the conic (2), then

$\frac{l}{-r_1} = e\cos(\pi + \theta_1) - 1$

or, $\frac{l}{-r_1} = -e\cos\theta_1 - 1$

or, $\frac{l}{r_1} = 1 + e\cos\theta_1 \longrightarrow (4)$

∴ From eqⁿ (2) and (4), it is clear that the given two equation are represent same conic.

12. The tangents at two points P and Q of a conic meet in T and S be the focus, if conic is parabola, then $ST^2 = SP \cdot SQ$. [2075, 2054]

⇒ Solution

Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be any two points on the conic,

$$\frac{l}{r} = 1 + e\cos\theta$$

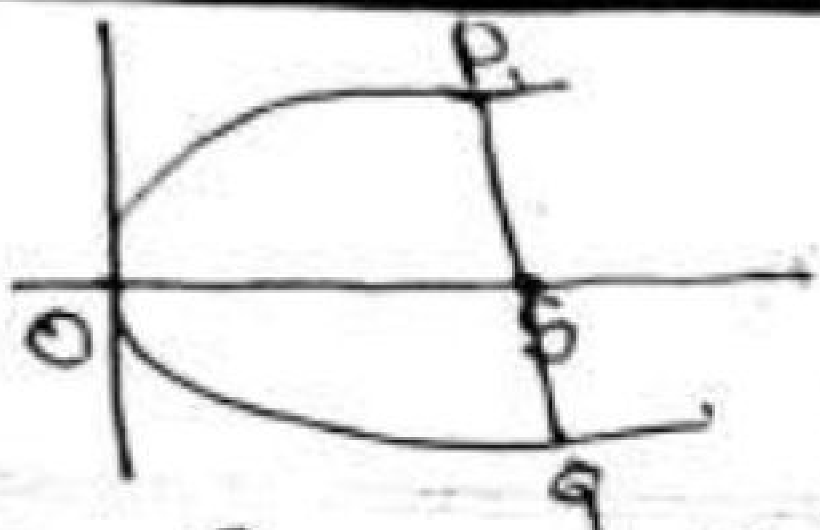
$$= 1 + \cos\theta \quad \text{Since } e = 1$$

$$\text{or, } \frac{l}{r} = 2\cos^2\frac{\theta}{2} \longrightarrow (1)$$

If T be the Point of intersection of two tangents then coordinates are

$$(r', \alpha) = \left[\frac{l}{\cos\left(\frac{\theta_2-\theta_1}{2}\right) + \cos\left(\frac{\theta_2+\theta_1}{2}\right)}, \frac{\theta_1+\theta_2}{2}\right]$$

$$r' = \frac{l}{\cos\left(\frac{\theta_2-\theta_1}{2}\right) + \cos\left(\frac{\theta_2+\theta_1}{2}\right)}$$



or, $\frac{l}{r'} = 2\cos\frac{\theta_2}{2} \cos\frac{\theta_1}{2}$

$\therefore \frac{l}{r'} = 2\cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \longrightarrow (2)$

and

$\alpha = \frac{\theta_1 + \theta_2}{2}$

Also, $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ lie in eqn. (1).

$\frac{l}{SP} = 1 + \cos\theta_1 = 2\cos^2\frac{\theta_1}{2}$

$\frac{l}{SQ} = 1 + \cos\theta_2 = 2\cos^2\frac{\theta_2}{2}$

$\frac{l}{SP} \cdot \frac{l}{SQ} = 4\cos^2\frac{\theta_1}{2} \cdot \cos^2\frac{\theta_2}{2}$

$\frac{l^2}{SP \cdot SQ} = \left(2\cos\frac{\theta_1}{2} \cdot \cos\frac{\theta_2}{2}\right)^2$

or, $\frac{l^2}{SP \cdot SQ} = \frac{l^2}{r'^2}$

or, $SP \cdot SQ = ST^2$ Proved

13. Prove that the locus of the middle points of the focal chords of the conic section is a conic section. [2054, 2064, 71]

⇒ Solution,

Let PSQ be a focal chord of a conic $\frac{l}{r} = 1 + e\cos\theta$, S being the focus.

Let α be the vectorial angle of P, the π+α be the vectorial angle of Q.

We know,

$\frac{l}{SP} = 1 + e\cos\alpha$ and

$\frac{l}{SQ} = 1 + e\cos(\pi+\alpha)$

Then,

$SP = \frac{l}{1+e\cos\alpha}$ and

$SQ = \frac{l}{1+e\cos(\pi+\alpha)} = \frac{l}{1-e\cos\alpha}$

Let $T(r', \alpha)$ be the middle point of the chord PSQ, then

$ST = r'$

Also,

$$ST = SP - TP = SP - \frac{SP + SQ}{2}$$

$$= \frac{SP - SQ}{2}$$

$$\therefore r' = \frac{l}{2}\left(\frac{1}{1+e\cos\alpha} - \frac{1}{1-e\cos\alpha}\right)$$

$$r' = \frac{l}{2}\left(\frac{1-e\cos\alpha-1-e\cos\alpha}{1-e^2\cos^2\alpha}\right)$$

$$= \frac{l}{2}\left(\frac{-2e\cos\alpha}{1-e^2\cos^2\alpha}\right)$$

The locus of $T(r', \alpha)$ is

$$r = \frac{l}{2} \cdot \frac{-2e\cos\theta}{1-e^2\cos^2\theta}$$

or, $r - re^2\cos^2\theta = -el\cos\theta$

or, $r^2 - r^2e^2\cos^2\theta + rel\cos\theta = 0 \longrightarrow (2)$

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Cartesian form of eq^n (1) is

$x^2+y^2-e^2x^2+elx=0 \longrightarrow (2)$

or, $(1-e^2)x^2+y^2+elx=0$,
Which ~~is~~ will be a parabola, ellipse and hyperbola according as

$e=1$, then eq^n (2) represents parabola.
$e>1$, then eq^n (2) represents hyperbola.
$e<1$, then eq^n (2) represents ellipse.