→ 1 long and 1 short question
7 + 4 = 11 marks

Chapter-1

Product of Three or four vectors

Scalar Triple product

Q.1 Define scalar triple product and prove that geometrically that the scalar triple product represents the volume of the parallelopiped. Also verify that in the scalar triple product position of dot and cross can be interchanged.

⇒ solution, [2054, 2057, 2055, 2066, 2067]

Scalar triple product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three given non-zero vectors in space. Then $\vec{b}\times\vec{c}$ is a vector perp. to the plane containing vectors $\vec{b}$ and $\vec{c}$. Again, $\vec{a}$ and $\vec{b}\times\vec{c}$ both being vectors, their dot product denoted by $\vec{a}\cdot(\vec{b}\times\vec{c})$ is known as scalar triple product. It is also denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$.

Therefore, the scalar product of two vectors, one of which being again a vector product of two vectors is a scalar and is known as the scalar triple product.

Geometrical interpretation of scalar triple product

Consider a parallelopied with three concurrent edges OA, OB and OC which represent in

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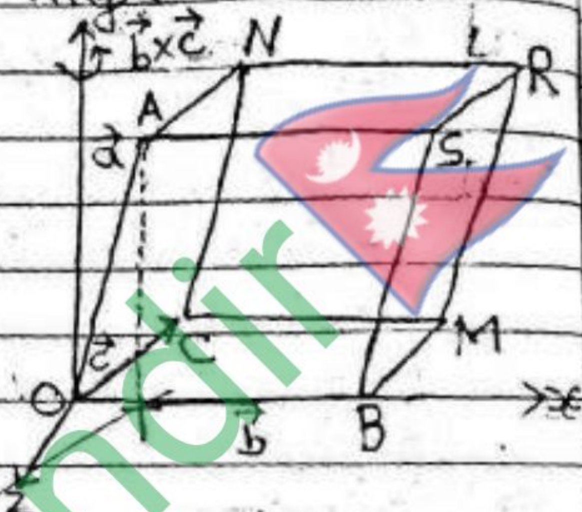
magnitude and direction of the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then $\vec{b}\times\vec{c}$ is the vector perpendicular to the plane of $\vec{OB}$ and $\vec{OC}$ whose magnitude is the area of the parallelogram OBMC.



Then $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of OB and OC whose magnitude is the area of parallelogram OBMC

Then,

$\vec{a}\cdot(\vec{b}\times\vec{c})$ = magnitude of $\vec{b}\times\vec{c}$ x projection of $\vec{a}$ on $(\vec{b}\times\vec{c})$

= Area of parallelogram OBMC (AT)

= Area of parallelogram OBMC (Height)

= Volume of parallelogram

= V

Similarly, we can show that

$\vec{b}\cdot(\vec{c}\times\vec{a}) = V$ and $\vec{c}\cdot(\vec{a}\times\vec{b}) = V$

Therefore,

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = V$

Hence, the scalar triple product represents the volume of parallelogram.

Next part:

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. Then we need to show that

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a\times\vec{b})\cdot\vec{c}$$

Let, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

and $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Then,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \longrightarrow (1)$$

Also,

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{c}\cdot(\vec{a}\times\vec{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\therefore (\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \longrightarrow (2)$$

Thus, from (1) and (2)

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

Hence in scalar triple product, the position of dot and cross can be interchanged.

Last part: If the scalar triple product is zero then,

(I) any two of the given vectors are equal and parallel

(II) all three vectors are coplanar.

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2. How many different type of products among three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ can be made? Give the geometrical meaning of $\vec{a} \times (\vec{b} \times \vec{c})$. Deduce the distributive law of cross product of two vectors by the help of scalar triple product. [2054, 2070, 2073]

$\Rightarrow$ Solution,

Combination of products among three vectors

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero given vectors in space, then the following six different kind of combinations of products can be made.

(I) $\vec{a} \times (\vec{b} \times \vec{c})$ (vector)
(II) $\vec{a} \cdot (\vec{b} \times \vec{c})$ (scalar)
(III) $\vec{a}\ (\vec{b} \times \vec{c})$ (undefined)
(IV) $\vec{a} \times (\vec{b} \cdot \vec{c})$ (vector)
(V) $\vec{a} \cdot (\vec{b} \cdot \vec{c})$ (meaningless)
(VI) $\vec{a}\ (\vec{b} \cdot \vec{c})$ (meaningless).

Geometrical meaning:

If $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero, non coplaner vectors in space. Then their vector triple product denoted by $\vec{a} \times (\vec{b} \times \vec{c})$ is perp. to both $\vec{a}$ and $(\vec{b} \times \vec{c})$.

So, $\vec{a} \times (\vec{b} \times \vec{c}) \cdot (\vec{b} \times \vec{c}) = 0$

or, $[\vec{a} \times (\vec{b} \times \vec{c})\ \vec{b}\ \vec{c}] = 0$

Hence, $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector coplaner with vectors

$\vec{b}$ and $\vec{c}$.

Next part

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ given three vectors in space.

Then, $\vec{a} \times (\vec{b} * \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

To prove this, let $\vec{r} = \vec{a} \times (\vec{b} + \vec{c}) - [(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})]$ (1)

Let $u \neq 0$ be any arbitrary vector. The taking dot product with $\vec{u}$ on both sides of eq^n (1)

$\vec{u} \cdot \vec{r} = \vec{u} \cdot [\vec{a} \times (\vec{b} + \vec{c}) - (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})]$

or, $\vec{u} \cdot \vec{r} = \vec{u} \cdot \vec{a} \times (\vec{b} + \vec{c}) - \vec{u} \cdot (\vec{a} \times \vec{b}) - \vec{u} \cdot (\vec{a} \times \vec{c})$

or, $\vec{u} \cdot \vec{r} = (\vec{u} \times \vec{a}) \cdot (\vec{b} + \vec{c}) - (\vec{u} \times \vec{a}) \cdot \vec{b} - (\vec{u} \times \vec{a}) \cdot \vec{c}$

[$\because$ In scalar triple product, the position of dot and cross can be interchanged.]

or, $\vec{u} \cdot \vec{r} = (\vec{u} \times \vec{a}) \cdot \{\vec{b} + \vec{c} - \vec{b} - \vec{c}\} = 0$

Since $\vec{u} \neq 0$, either $\vec{r} = 0$ or $\vec{u}$ is perpendicular to $\vec{r}$, so we have supposed that $\vec{u}$ is any arbitrary vector so it is not perpendicular to $\vec{r}$.

So, $\vec{r} = 0$

Hence from eq^n (1)

$\vec{a} \times (\vec{b} * \vec{c}) - (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = 0$

$\therefore \vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

Hence, proved

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Q.3. When will the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in space be coplaner? Show that $\vec{a}, \vec{b}, \vec{c}$ are coplaner if and only if $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are coplaner. [2071]

If $[\vec{a}\,\vec{b}\,\vec{c}] = 0$, then show that $[\vec{a}\times\vec{b}\;\; \vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}] = 0$ [2055]

⇒ Solution,

Condition of coplaner

The given vector $\vec{a}, \vec{b}, \vec{c}$ in space are said to be coplaner if scalar triple product betn them is zero i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

Next part

$$[\vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}\;\; \vec{a}\times\vec{b}] = (\vec{b}\times\vec{c}) \cdot [(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$$

$$= (\vec{b}\times\vec{c}) \cdot \{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b}\}$$

$$= (\vec{b}\times\vec{c}) \cdot \{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a}\}$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\,[\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]^2$$

$$\therefore [\vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}\;\; \vec{a}\times\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 \longrightarrow (1)$$

If the given vectors $\vec{a}, \vec{b}, \vec{c}$ are coplaner then $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

So, from eqn. (1),

$[\vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}\;\; \vec{a}\times\vec{b}] = 0$

Hence, $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$ are coplaner.

Suppose the given vectors $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$ be coplaner; then

$[\vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}\;\; \vec{a}\times\vec{b} = 0]$, or. $[\vec{a}\,\vec{b}\,\vec{c}]^2 = 0$, or, $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

4. Prove that the scalar triple product of three non-zero vectors is zero when two of the vectors are equal and parallel. [2060, 2068, 2071]

⇒ Solution,

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the three non-zero given vectors, then their scalar triple product is $[\vec{a}\,\vec{b}\,\vec{c}]$

(I) If any two vectors are equal i.e. $\vec{c} = \vec{a}$, Then

$$[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{a}] = \vec{a}\cdot(\vec{b}\times\vec{a}) = (\vec{b}\times\vec{a})\cdot\vec{a}$$
$$= \vec{b}\cdot(\vec{a}\times\vec{a}) = 0$$

(II) If any two vectors are parallel

Let $\vec{a}$ and $\vec{b}$ are parallel. Then $\vec{a}$ can be expressed as the scalar multiple of $\vec{b}$ i.e. $\vec{a} = k\vec{b}$ where k be a scalar.

Now,

$$[\vec{a}\,\vec{b}\,\vec{c}] = [k\vec{b}\,\vec{b}\,\vec{c}] = k[\vec{b}\,\vec{b}\,\vec{c}]$$
$$= k\vec{b}\cdot(\vec{b}\times\vec{c})$$
$$= k[(\vec{b}\times\vec{b})\cdot\vec{c}]$$
$$= 0$$

Hence, proved

5. Prove that scalar triple product of three vectors is zero if,
(I) Two of the vectors are parallel
(II) Three vectors are coplaner. [2064]

⇒ Solution,

(I) To show $[\vec{a}\vec{b}\vec{c}] = 0$, if any two vector are parallel.

Let $\vec{a}$ and $\vec{b}$ be parallel. Then $\vec{a}$ can can be expressed as the scalar multiple of $\vec{b}$ i.e. $\vec{a} = k\vec{b}$; where k is scalar.

Now,

$$[\vec{a}\vec{b}\vec{c}] = [k\vec{b}\vec{b}\vec{c}]$$
$$= k[\vec{b}\vec{b}\vec{c}]$$
$$= k\vec{b}\cdot(\vec{b}\times\vec{c})$$
$$= k\{(\vec{b}\times\vec{b})\cdot\vec{c}\}$$
$$= 0$$

Hence, proved.

(II) We need to show that scalar triple product ~~amoun~~ among three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ i.e. $[\vec{a}\vec{b}\vec{c}]$ is zero if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Necessary condition:

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three coplanar vectors, then we need to show that $[\vec{a}\vec{b}\vec{c}] = 0$

We know that

$\vec{b}\times\vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$. Since $\vec{a}, \vec{b}, \vec{c}$ are coplaner, $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$. Hence, dot product between them is zero

i.e. $(\vec{b} \times \vec{c}) \cdot \vec{a} = 0$

or, $[\vec{b}\,\vec{c}\,\vec{a}] = 0$

or, $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

The sufficient conditions:

Let $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

or, $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

So, $\vec{a}$ and $(\vec{b} \times \vec{c})$ are perpendicular to each other and $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane, $\vec{b}$ and $\vec{c}$. Hence $\vec{a}$ should be in the plane of $\vec{b}$ and $\vec{c}$. Thus the given vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplaner.

6. Prove the following

(a) $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$

(b) If $\vec{a} + \vec{b} + \vec{c} = 0$, then $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

[2062, 2066, 2068]

$\Rightarrow$ Solution,

(a) $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$

i.e. $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = a^2b^2$

Now,

$(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) + (\vec{a} \cdot \vec{b})^2$

$= [|\vec{a}||\vec{b}| \sin\theta\, \hat{n}] \cdot [|\vec{a}||\vec{b}| \sin\theta\, \hat{n}] + (\vec{a} \cdot \vec{b})^2$

$= (ab\sin\theta\, \hat{n}) \cdot (ab\sin\theta\, \hat{n}) + (ab\cos\theta)^2$

$[\because \vec{a} \cdot \vec{b} = ab\cos\theta]$

$[\vec{a}\times\vec{a}=0]$

$= a^2b^2\sin^2\theta\,(\hat{n}\cdot\hat{n}) + a^2b^2\cos^2\theta$

$= a^2b^2(\sin^2\theta+\cos^2\theta)$ $[\because \hat{n}\cdot\hat{n}=1]$

$= a^2b^2$ $[\because \sin^2\theta+\cos^2\theta=1]$

Hence, proved

(b) Here, $\vec{a}+\vec{b}+\vec{c}=0 \longrightarrow (1)$

Taking cross product with $\vec{a}$ on both sides

$\vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{a}\times 0$

or, $0+\vec{a}\times\vec{b}-\vec{c}\times\vec{a}=0$

or, $\vec{a}\times\vec{b}=\vec{c}\times\vec{a} \longrightarrow (2)$

Again, Taking cross product with $\vec{b}$ on both sides,

$\vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=\vec{b}\times 0$

$-\vec{a}\times\vec{b}+0+\vec{b}\times\vec{c}=0$

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c} \longrightarrow (3)$

Combining eqn (2) and (3), we got

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

Hence proved

7. Show that $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]=2[\vec{a}\,\vec{b}\,\vec{c}]$

[2067, 2070, 2075]

$\Rightarrow$ Solution,

We have

L.H.S

$[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]$

$=(\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$=(\vec{a}+\vec{b})\cdot[\vec{b}\times(\vec{c}+\vec{a})+\vec{c}(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$= (\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= \vec{a}\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]+\vec{b}\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})$
$+\vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{a}]+[\vec{b}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{b}\,\vec{a}]+[\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]+0+0+0+0+[\vec{a}\,\vec{b}\,\vec{c}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

Hence, $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

Proved

8 If $\vec{a},\vec{b},\vec{c}$ are non coplanar, then show $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also non coplaner.

⇒ solution,

Given, $\vec{a},\vec{b}\,\vec{c}$ are non coplanar vectors in space, then

$[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$

Now we prove, $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are non coplanar.

So, $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b})\cdot[\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$

$= (\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= \cancel{(\vec{a}+\vec{b})}\;\vec{a}\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]+\vec{b}\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$

$= \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})$
$+\vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}]+[\vec{a}\,\vec{b}\,\vec{a}]+[\vec{a}\,\vec{c}\,\vec{a}]+[\vec{b}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{b}\,\vec{a}]+$
$[\vec{b}\,\vec{c}\,\vec{a}]$

$= 2[\vec{a}\,\vec{b}\,\vec{c}]$

Since, $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$

Hence, $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are non coplanar.

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Vector triple product

9. How do you define vector triple product? Interpret it geometrically. If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors then show that
$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{c}\times(\vec{a}\times\vec{b}) + \vec{b}\times(\vec{c}\times\vec{a}) = 0$

⇒ Solution, [2056]

(I) Vector triple product

Let $\vec{a}, \vec{b}, \vec{c}$ be the three non-zero vectors in space. Then $\vec{b}\times\vec{c}$ is a vector normal to the plane of $\vec{b}$ and $\vec{c}$. Again since $\vec{a}$ and $\vec{b}\times\vec{c}$ both are vectors, their vector product denoted by $\vec{a}\times(\vec{b}\times\vec{c})$ is known as vector triple product.

(II) Geometrical meaning

Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero, non coplaner vectors. Then their vector triple product denoted by $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to both $\vec{a}$ and $(\vec{b}\times\vec{c})$.

So, $\vec{a}\times(\vec{b}\times\vec{c}) \cdot (\vec{b}\times\vec{c}) = 0$

or, $[\vec{a}\times(\vec{b}\times\vec{c})\ \vec{b}\ \vec{c}] = 0$

Hence, $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector coplanar with $\vec{b}$ and $\vec{c}$.

Last part:

We have to prove that

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

L.H.S $= \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}$

$= 0$ Hence proved

10. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$ then verify that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ [2060, 2068 new]

⇒ Solution,

$\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$

Then, $\vec{b}\times\vec{c} = (2\vec{i} + \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$= -3\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{i} - 2\vec{j} + \vec{k}) \times (-3\vec{i} + 3\vec{j} + 3\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

$\therefore \vec{a}\times(\vec{b}\times\vec{c}) = -9\vec{i} - 6\vec{j} - 3\vec{k} \longrightarrow (1)$

Again,

$\vec{a}\cdot\vec{b} = (\vec{i} - 2\vec{j} + \vec{k})\cdot(2\vec{i} + \vec{j} + \vec{k})$

$= 2 - 2 + 1 = 1$

Also, $\vec{a}\cdot\vec{c} = (\vec{i} - 2\vec{j} + \vec{k})(\vec{i} + 2\vec{j} - \vec{k})$

$= 1 - 4 - 1 = -4$

Now,

~~Since $\vec{a}\times(\vec{b}\times\vec{c})$ is the vector triple product~~

- Since $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{b}\times\vec{c})$ are perpendicular; their dot product must be equal to zero. So

$[\vec{a}\times(\vec{b}\times\vec{c}) \cdot \text{and } (\vec{b}\times\vec{c})] = 0$

or, $[\vec{a}\times(\vec{b}\times\vec{c})\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow a\times(\vec{b}\times\vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$.

Vector triple product vanishes if any two coplanar vectors are equal or parallel.

Last part

we need to show that,

$[\vec{\alpha}+\vec{\beta}\ \ \vec{\beta}+\vec{\gamma}\ \ \vec{\gamma}+\vec{\alpha}] = 2[\vec{\alpha}\vec{\beta}\vec{\gamma}]$

L.H.S.

$[\vec{\alpha}+\vec{\beta}\ \ \vec{\beta}+\vec{\gamma}\ \ \vec{\gamma}+\vec{\alpha}]$

$= (\vec{\alpha}+\vec{\beta})\cdot[(\vec{\beta}+\vec{\gamma})\times(\vec{\gamma}+\vec{\alpha})]$

$= (\vec{\alpha}+\vec{\beta})\cdot[\vec{\beta}\times(\vec{\gamma}+\vec{\alpha})+\vec{\gamma}\times(\vec{\gamma}+\vec{\alpha})]$

$= (\vec{\alpha}+\vec{\beta})\cdot[(\vec{\beta}\times\vec{\gamma})+(\vec{\beta}+\vec{\alpha})+(\vec{\gamma}\times\vec{\alpha})]$

$= \vec{\alpha}\cdot[(\vec{\beta}\times\vec{\gamma})+(\vec{\beta}\times\vec{\alpha})+(\vec{\gamma}\times\vec{\alpha})]+\vec{\beta}\cdot[(\vec{\beta}\times\vec{\gamma})+(\vec{\beta}\times\vec{\alpha})+(\vec{\gamma}\times\vec{\alpha})]$

$= \vec{\alpha}\cdot\vec{\beta}\times\vec{\gamma}+\vec{\alpha}\cdot\vec{\beta}\times\vec{\alpha}+\vec{\alpha}\cdot\vec{\gamma}\times\vec{\alpha}+\vec{\beta}\cdot\vec{\beta}\times\vec{\gamma}+\vec{\beta}\cdot\vec{\beta}\times\vec{\alpha}$
$\quad + \vec{\beta}\cdot\vec{\gamma}\times\vec{\alpha}$

$= [\vec{\alpha}\vec{\beta}\vec{\gamma}]+0+0+0+0+[\vec{\beta}\ \vec{\gamma}+\vec{\alpha}]$

$= [\vec{\alpha}\vec{\beta}\vec{\gamma}]+[\vec{\alpha}\vec{\beta}\vec{\gamma}]$

$= 2[\vec{\alpha}\vec{\beta}\vec{\gamma}]$

Hence, R.H.S proved

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$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -4(2\vec{i}+\vec{j}+\vec{k}) - 1(\vec{i}+2\vec{j}-\vec{k})$

or, $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -8\vec{i}-4\vec{j}-4\vec{k}-1\vec{i}-2\vec{j}+1\vec{k}$

or, $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -9\vec{i}-6\vec{j}-3\vec{k} \longrightarrow (2)$

Hence, from eqⁿ (1) & (2), we get

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -9\vec{i}-6\vec{j}-3\vec{k}$

Hence proved

11 Define the vector triple product and show that $\vec{a}\times(\vec{b}\times\vec{c})$ lies in the plane of $\vec{b}$ and $\vec{c}$. Also state when does vector triple product vanish. prove the relation $[\vec{\alpha}+\vec{\beta}\ \ \vec{\beta}+\vec{\gamma}\ \ \vec{\gamma}+\vec{\alpha}] = 2[\vec{\alpha}\vec{\beta}\vec{\gamma}]$ [2062, 2073]

⇒ Solution,

(i) Vector triple product.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors in space. Then $\vec{b}\times\vec{c}$ is a vector normal to the plane of $\vec{b}$ and $\vec{c}$. Again, since $\vec{a}$ and $\vec{b}\times\vec{c}$ both are vectors, their product denoted by $\vec{a}\times(\vec{b}\times\vec{c})$ is a is vector which is known as vector triple product.

Next part

~~We need t~~

Here, $\vec{a}\times(\vec{b}\times\vec{c})$ is the vector triple product among three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Then $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$. So $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector perpendicular to the plane of $\vec{a}$ and $(\vec{b}\times\vec{c})$.

12. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$, find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, $\vec{b}$ and $\vec{c}$ being non parallel.
Prove that: $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$
[2059, 2061, 2067, 2071, 2072]

⇒ Solution,
Here, $\vec{a}, \vec{b}, \vec{c}$ are unit vectors
$\therefore |\vec{a}| = 1, |\vec{b}| = 1\ |\vec{c}| = 1$
Given condition is
$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$

or, $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{2}\vec{b}$

or, $(\vec{a}\cdot\vec{c} - \frac{1}{2})\vec{b} + (-(\vec{a}\cdot\vec{b})\vec{c}) = 0$
Since, $\vec{b}$ and $\vec{c}$ being non-parallel
$\therefore$ either, $\vec{a}\cdot\vec{c} - \frac{1}{2} = 0$, or, $\vec{a}\cdot\vec{b} = 0$

Now, Case-1: $\vec{a}\cdot\vec{c} - \frac{1}{2} = 0$

or $\vec{a}\cdot\vec{c} = \frac{1}{2}$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then
$|\vec{a}|\,|\vec{c}| \cos\theta = \frac{1}{2}$

or $\cos\theta = \frac{1}{2}$ or, $\cos\theta = \cos\frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3}$

Case - II : $\vec{a}\cdot\vec{b} = 0$

If ϕ be the angle between $\vec{a}$ and $\vec{b}$, then

$|\vec{a}||\vec{b}|\cos\phi = 0$

or, $\cos\phi = 0$

or, $\cos\phi = \cos\frac{\pi}{2}$

$\therefore \phi = \frac{\pi}{2}$

Hence, the angle betn $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$ and $\vec{a}$ and $\vec{c}$ is $\frac{\pi}{3}$.

Next part,

L.H.S $= [\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$

$= (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$

$= (\vec{a}\times\vec{b})\cdot[\vec{u}\times(\vec{c}\times\vec{a})]$ $[\because u = \vec{b}\times\vec{c}]$

$= (\vec{a}\times\vec{b})\cdot[(\vec{u}\cdot\vec{a})\vec{c} - (\vec{u}\cdot\vec{c})\vec{a}]$

By definition of vector triple product

$= (\vec{a}\times\vec{b})\cdot([\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - [\vec{b}\,\vec{c}\,\vec{c}]\vec{a})$

$= (\vec{a}\times\vec{b})\cdot\{[\vec{a}\,\vec{b}\,\vec{c}]\vec{c} - 0\}$

$= (\vec{a}\times\vec{b})\cdot\vec{c}\,[\vec{a}\,\vec{b}\,\vec{c}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]^2$

Hence Proved

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13. Find the expression for $\vec{a} \times (\vec{b} \times \vec{c})$.
[2063, 2067, 2071, 2071 new, 2073]

$\Rightarrow$ Solution,

Let, the given three vectors be

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

And $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Now,

$$\vec{b} \times \vec{c} = (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \times (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\vec{i} + (b_3c_1 - b_1c_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$$

$\therefore \vec{a} \times (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= [a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3]\vec{i} + [a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1]\vec{j} + [a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2]\vec{k}$$

$= [b_1(a_1c_1 + a_2c_2 + a_3c_3) - (a_1b_1 + a_2b_2 + a_3b_3)c_1]\vec{i}$
$+ [b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)]\vec{j}$
$+ [b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3)]\vec{k}$

Adding and subtracting $a_1b_1c_1$ in first, $a_2b_2c_2$ in second, $a_3b_3c_3$ in third respectively.

$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$

$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Since, $\vec{a}\cdot\vec{c} = a_1c_1 + a_2c_2 + a_3c_3$ and
$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Hence, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

14. Prove that
(i) $b^2\vec{a} = (\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$.
(ii) $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$
[2072 old, 2075 New]

⇒ Solution,
(I) R.H.S. $= (\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$
$= (\vec{a}\cdot\vec{b})\vec{b} + (\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}$
$= (\vec{a}\cdot\vec{b})\vec{b} + b^2\vec{a} - (\vec{a}\cdot\vec{b})\vec{b}$
$= b^2\vec{a}$

Hence, proved

(II) L.H.S. $= \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$
$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$
$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b} = 0$ proved

Product of four vectors.

15. Define vector product of four vectors and determine the expression for this product. Also, obtain the linear relation between them. [2054]

How do you define scalar and vector triple product of four vectors. Find the expression for scalar product of four vectors. Determine the linear connecting four given vectors $\vec{a}, \vec{b}, \vec{c}$ and where $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ [2065, 2071 old]

⇒ Solution,

(i) Scalar product of four vectors:

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be four given vectors. Then $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are the vectors normal to the plane containing $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Since $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ both are vectors, then the scalar product between them denoted by $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$, is a scalar and this product of four vectors.

(ii) Vector product of four vectors:

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be four vectors. Then $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ are the vectors normal to the plane containing $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively since $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$ both are vectors, then the vector product between them denoted by $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ is called vector product of four vectors.

Date ______
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Expression for $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

Let, $\vec{a}\times\vec{b} = \vec{u}$

Then, $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

$= \vec{u}\cdot(\vec{c}\times\vec{d})$

$= (\vec{u}\times\vec{c})\cdot\vec{d}$

In scalar triple product; dot and cross can be interchanged.

$= \{(\vec{a}\times\vec{b})\times\vec{c}\}\cdot\vec{d}$

$= -\{\vec{c}\times(\vec{a}\times\vec{b})\}\cdot\vec{d}$

$= -\{(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\}\cdot\vec{d}$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Which is the required expression for scalar product of four vectors.

Expression for $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

Let $\vec{c}\times\vec{d} = \vec{u}$

Then, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$= (\vec{a}\times\vec{b})\times\vec{u}$

$= -\vec{u}\times(\vec{a}\times\vec{b})$ $\quad (\because \vec{a}\times\vec{b} = -\vec{b}\times\vec{a})$

$= -[(\vec{u}\cdot\vec{b})\vec{a} - (\vec{u}\cdot\vec{a})\vec{b}]$

$= -[(\vec{b}\cdot\vec{u})\vec{a} - (\vec{a}\cdot\vec{u})\vec{b}]$

$= -[\vec{b}\cdot\vec{c}\times\vec{d}]\vec{a} + [\vec{a}\cdot\vec{c}\times\vec{d}]\vec{b}$

$= -[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} + [\vec{a}\,\vec{c}\,\vec{d}]\vec{b}$

$\therefore (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} \longrightarrow (I)$

Similarly,

Let $\vec{a}\times\vec{b} = \vec{v}$

Then,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{v}\times(\vec{c}\times\vec{d})$$
$$= (\vec{v}\cdot\vec{d})\vec{c} - (\vec{v}\cdot\vec{c})\vec{d}$$
$$= [(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c} - [(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d}$$

$\therefore (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$ —— (2)

Combining eq^n (1) and (2)

$$[\vec{a}\vec{c}\vec{d}]\vec{b} - [\vec{b}\vec{c}\vec{d}]\vec{a} = [\vec{a}\vec{b}\vec{d}]\vec{c} - [\vec{a}\vec{b}\vec{c}]\vec{d}$$

or, $[\vec{a}\vec{b}\vec{c}]\vec{d} = [\vec{b}\vec{c}\vec{d}]\vec{a} - [\vec{a}\vec{c}\vec{d}]\vec{b} + [\vec{a}\vec{b}\vec{d}]\vec{c}$

or, $[\vec{a}\vec{b}\vec{c}]\vec{d} = [\vec{b}\vec{c}\vec{d}]\vec{a} + [\vec{c}\vec{a}\vec{d}]\vec{b} + [\vec{a}\vec{b}\vec{d}]\vec{c}$

or, $\vec{d} = \dfrac{[\vec{b}\vec{c}\vec{d}]\vec{a} + [\vec{c}\vec{a}\vec{d}]\vec{b} + [\vec{a}\vec{b}\vec{d}]\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$

$$\vec{d} = \frac{[\vec{b}\vec{c}\vec{d}]\vec{a}}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{c}\vec{a}\vec{d}]\vec{b}}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{a}\vec{b}\vec{d}]\vec{c}}{[\vec{a}\vec{b}\vec{c}]} \neq 0$$

Which is the required linear relation between four vectors.

16. Show that $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = [\vec{a}\vec{b}\vec{c}]\vec{c}$. Using this relation prove that $[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = [\vec{a}\vec{b}\vec{c}]^2$ [2054]

$\Rightarrow$ Solution,

First part.

Let $\vec{b}\times\vec{c} = \vec{u}$ then

$$(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{u}\times(\vec{c}\times\vec{a})$$
$$= (\vec{u}\cdot\vec{a})\vec{c} - (\vec{u}\cdot\vec{c})\vec{a}$$
$$= [(\vec{b}\times\vec{c})\cdot\vec{a}]\vec{c} - [(\vec{b}\times\vec{c})\cdot\vec{c}]\vec{a}$$
$$= [\vec{b}\vec{c}\vec{a}]\vec{c} - [\vec{b}\vec{c}\vec{c}]\vec{a}$$
$$= [\vec{a}\vec{b}\vec{c}]\vec{c}$$

Proved

Next part

$[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = (\vec{b}\times\vec{c})\cdot[(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$

$= [(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]\cdot(\vec{a}\times\vec{b})$

Since, in scalar triple product, position of dot and cross product can be interchanged.

$= \{[\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - [\vec{b}\,\vec{c}\,\vec{c}]\vec{a}\}\cdot(\vec{a}\times\vec{b})$

$= [\vec{a}\,\vec{b}\,\vec{c}]\vec{c}\cdot(\vec{a}\times\vec{b})$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{c}\cdot(\vec{a}\times\vec{b})]$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]^2$

Hence, $[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

Proved

17. What do you mean by scalar product of four vectors? Show that

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0$

[2058]

⇒ Solution,

L.H.S

$(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

$= \begin{vmatrix} \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{d} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{d} \\ \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

$= (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d}) - (\vec{c}\cdot\vec{a})(\vec{b}\cdot\vec{d}) + (\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d}) - (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})$
$+ (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$

$= (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) + (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d}) -$
$(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) + (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$

$= 0$ proved

19. Define vector product of four vectors. prove that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})$
$= -2[\vec{a}\,\vec{c}\,\vec{b}]\vec{d}$ [2071 old]

$\Rightarrow$ Solution,

L.H.S. $= (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a})\times(\vec{b}\times\vec{d})$

$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} + [\vec{b}\,\vec{c}\,\vec{d}]\vec{c} - [\vec{b}\,\vec{c}\,\vec{a}]\vec{d} + [\vec{c}\,\vec{b}\,\vec{d}]\vec{a}$
$- [\vec{a}\,\vec{b}\,\vec{d}]\vec{c}$

$= -2[\vec{a}\,\vec{c}\,\vec{b}]\vec{d}$

Since, $[\vec{c}\,\vec{b}\,\vec{d}] = -[\vec{b}\,\vec{c}\,\vec{d}]$ and $[\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]$

20. Prove that $[\vec{l}\,\vec{m}\,\vec{n}][\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$

[2070 old]

$\Rightarrow$ Solution,

Every vector can be expressed in terms of three mutually perpendicular unit vectors $\vec{i}, \vec{j}, \vec{k}$.

Let, $\vec{l} = l_1\vec{i} + l_2\vec{j} + l_3\vec{k}$,
$\vec{m} = m_1\vec{i} + m_2\vec{j} + m_3\vec{k}$,
$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$

and, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$
$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Then, $[\vec{l}\,\vec{m}\,\vec{n}] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$,

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & b_2 & c_3 \end{vmatrix}$$

$$[\vec{l}\,\vec{m}\,\vec{n}]\,[\vec{a}\,\vec{b}\,\vec{c}]$$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1a_1 + l_2a_2 + l_3a_3 & l_1b_1 + l_2b_2 + l_3b_3 & l_1c_1 + l_2c_2 + l_3c_3 \\ m_1a_1 + m_2a_2 + m_3a_3 & m_1b_1 + m_2b_2 + m_3b_3 & m_1c_1 + m_2c_2 + m_3c_3 \\ n_1a_1 + n_2a_2 + n_3a_3 & n_1b_1 + n_2b_2 + n_3b_3 & n_1c_1 + n_2c_2 + n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$$

$$\therefore [\vec{l}\,\vec{m}\,\vec{n}]\,[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$$

Proved

2077/09/15

Reciprocal System of vectors

20. Define reciprocal system of vectors prove that

(I) $[\vec{a}'\vec{b}'\vec{c}'] = \dfrac{1}{[\vec{a}\vec{b}\vec{c}]}$.

(II) $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$

(III) $\vec{a}\cdot\vec{b}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = 0$

Where $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

[2055, 2059, 2061, 2068, 2069, 2075, 2054]

⇒ solution,

Reciprocal system of vectors:

Let $\vec{a}, \vec{b}, \vec{c}$ be three non coplanar vectors in space such that $[\vec{a}\vec{b}\vec{c}] \neq 0$. Then the three vectors $\vec{a}', \vec{b}'$ and $\vec{c}'$ defined by the relation.

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]},\ \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

Which are perpendicular to the plane containing $\vec{b}, \vec{c}$; $\vec{c}, \vec{a}$; and $\vec{a}, \vec{b}$ respectively. It is known as reciprocal system of vectors to $\vec{a}, \vec{b}$ and $\vec{c}$.

(I) Since $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors, then

$$\vec{a}'\cdot(\vec{b}'\times\vec{c}') = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}\cdot\left[\frac{(\vec{c}\times\vec{a})}{[\vec{a}\vec{b}\vec{c}]}\times\frac{(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]}\right]$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}\cdot\left(\frac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]^2}\right.$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \cdot \left(\frac{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} \right)$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \cdot \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$= \frac{(\vec{b}\times\vec{c})\cdot\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$= \frac{[\vec{b}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\therefore [\vec{a}'\,\vec{b}'\,\vec{c}'] = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

(II) Since, $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

So, $\vec{a}\cdot\vec{a}' = \vec{a}\cdot\dfrac{(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\vec{a}\cdot\vec{a}' = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$

Similarly, $\vec{b}\cdot\vec{b}' = 1$, $\vec{c}\cdot\vec{c}' = 1$

$\therefore \vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$

Hence, proved

(III) Since, $\vec{b'} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$

So, $\vec{a}\cdot\vec{b'} = \vec{a}\cdot\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$

$\therefore \vec{a}\cdot\vec{b'} = \dfrac{[a\vec{c}\vec{a}]}{[\vec{a}\vec{b}\vec{c}]} = 0$

Similarly
$\vec{b}\cdot\vec{c'} = \vec{c}\cdot\vec{a'} = 0$

21. Find a set of vectors reciprocal system to the set of vectors $-\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}+\vec{k}$, and $\vec{i}+\vec{j}-\vec{k}$. [2057]

⇒ Solution,

Let $\vec{a} = -\vec{i}+\vec{j}+\vec{k}$, $\vec{b} = \vec{i}-\vec{j}+\vec{k}$, $\vec{c} = \vec{i}+\vec{j}-\vec{k}$.

Suppose the reciprocal system to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are $\vec{a'}$, $\vec{b'}$ and $\vec{c'}$ respectively. Then,

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1)+1(-1+1)+1(-1+1)$$

$$= 4$$

$$\vec{b}\times\vec{c} = (\vec{i}-\vec{j}+\vec{k})\times(\vec{i}+\vec{j}-\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 2\vec{j}+2\vec{k}$$

$$\vec{c}\times\vec{a} = (\vec{i}+\vec{j}-\vec{k})\times(-\vec{i}+\vec{j}+\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= 2\vec{i} + 2\vec{k}$$

and $\vec{a}\times\vec{b} = (-\vec{i}+\vec{j}+\vec{k})\times(\vec{i}-\vec{j}+\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 2\vec{i} + 2\vec{j}$$

Here by definition of reciprocal system of vectors,

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]} = \frac{2\vec{j}+2\vec{k}}{4} = \frac{1}{2}(\vec{j}+\vec{k})$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]} = \frac{2\vec{i}+2\vec{k}}{4} = \frac{1}{2}(\vec{i}+\vec{k})$$

and

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]} = \frac{2\vec{i}+2\vec{j}}{4} = \frac{1}{2}(\vec{i}+\vec{j})$$

P.T.O.

22. If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are the reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$ then show that, $[\vec{\alpha}\vec{\beta}\vec{\gamma}][\vec{a}\vec{b}\vec{c}] = 1$

[BSc 2063]

$\Rightarrow$ Solution,

Since $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are the reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then,

$$\vec{\alpha} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}, \quad \vec{\beta} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}, \quad \vec{\gamma} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

$$\vec{\beta}\times\vec{\gamma} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]} \times \frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

$$= \frac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]^2}$$

$$= \frac{[\vec{c}\vec{a}\vec{b}]\vec{a} - [\vec{c}\vec{a}\vec{a}]\vec{b}}{[\vec{a}\vec{b}\vec{c}]^2}$$

$$= \frac{[\vec{a}\vec{b}\vec{c}]\vec{a}}{[\vec{a}\vec{b}\vec{c}]^2}$$

$$\therefore \vec{\beta}\times\vec{\gamma} = \frac{\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$$

Now,

$$\vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma}) = \frac{(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]} \cdot \frac{\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$$

$$\text{or, } [\vec{\alpha}\vec{\beta}\vec{\gamma}] = \frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]^2}$$

$$\therefore [\vec{\alpha}\vec{\beta}\vec{\gamma}] = \frac{1}{[\vec{a}\vec{b}\vec{c}]}$$

Hence, $[\vec{\alpha}\vec{\beta}\vec{\gamma}]\cdot[\vec{a}\vec{b}\vec{c}] = 1$ Proved

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Q. If $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ then prove that $\vec{a}, \vec{b}, \vec{c}$ be the reciprocal system to $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$. Using this relation prove that $\vec{a}' \cdot \vec{a} = \vec{b}' \cdot \vec{b} = \vec{c}' \cdot \vec{c} = 1$.
[2065]

$\Rightarrow$ Solution.

Since, $\vec{a}', \vec{b}', \vec{c}'$ be the reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then,

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

We need to show that $\vec{a}, \vec{b}, \vec{c}$ are the reciprocal system to $\vec{a}', \vec{b}', \vec{c}'$

i.e. $$\vec{a} = \frac{\vec{b}' \times \vec{c}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}, \quad \vec{b} = \frac{\vec{c}' \times \vec{a}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$$

$$\vec{c} = \frac{\vec{a}' \times \vec{b}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$$

Now

$$\vec{b}' \times \vec{c}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \times \frac{\vec{a} \times \vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$= \frac{[(\vec{c} \times \vec{a}) \cdot \vec{b}]\,\vec{a} - [(\vec{c} \times \vec{a}) \cdot \vec{a}]\,\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$= \frac{[\vec{c}\,\vec{a}\,\vec{b}]\,\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\,\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$= \frac{[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$$

$$\therefore \vec{b}'\times\vec{c}' = \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Dividing both sides by $[\vec{a}'\,\vec{b}'\,\vec{c}']$, we get

$$\frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']} = \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}']}$$

$$= \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{a}'\cdot(\vec{b}'\times\vec{c}')}$$

$$= \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]\left\{\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right\}\cdot\left\{\frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right\}}$$

$$= \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\cdot\frac{1}{\frac{(\vec{b}\times\vec{c})\cdot\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2}}$$

$$= \frac{\vec{a}\,[\vec{a}\,\vec{b}\,\vec{c}]^2}{[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']} = \vec{a},$$

Similarly,

$$\frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']} = \vec{b}, \text{ and } \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\,\vec{b}'\,\vec{c}']} = \vec{c}.$$

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Next - part

$$\vec{a}\cdot\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\cdot\vec{a} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$$

Also, similarly, $\vec{b}'\cdot\vec{b} = \vec{c}'\cdot\vec{c} = 1$

Hence, proved

24. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors, then prove that $\vec{a}\cdot\vec{a}' + \vec{b}\cdot\vec{b}' + \vec{c}\cdot\vec{c}' = 3$ [2069 new]

$\Rightarrow$ solution,

since, $\vec{a}', \vec{b}', \vec{c}'$ are the reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ then,

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Now,

$$\vec{a}\cdot\vec{a}' = \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$$

Similarly, $\vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$

Hence, $\vec{a}\cdot\vec{a}' + \vec{b}\cdot\vec{b}' + \vec{c}\cdot\vec{c}' = 1+1+1$
$= 3$ Proved

25. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors defined by the relation

$\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$ then

prove that $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$. [2064]

⇒ Solution,

Since, $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$

L.H.S.

$\vec{a}\times\vec{a}' + \vec{b}\cdot\vec{b}' + \vec{c}\cdot\vec{c}'$

$= \vec{a}\times\dfrac{(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]} + \vec{b}\times\dfrac{(\vec{c}\times\vec{a})}{[\vec{a}\vec{b}\vec{c}]} + \vec{c}\times\dfrac{(\vec{a}\times\vec{b})}{[\vec{a}\vec{b}\vec{c}]}$

$= \dfrac{(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{c} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$

$= \dfrac{0}{[\vec{a}\vec{b}\vec{c}]}$

$= 0$

Hence, $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$

R.H.S. proved

26. Show that any vector $\vec{r}$ may be expressed as, $\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non coplanar vectors. 27.

[2070, Y.1]

⇒ Solution,

Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are non coplanar vectors. Hence any vector $\vec{r}$ can be expressed as the linear combination of $\vec{a}, \vec{b}, \vec{c}$.

So, let $\vec{r} = l\vec{a} + m\vec{b} + n\vec{c} \longrightarrow (1)$

where l, m, n be the scalars to be determined. Multiplying scalarly both sides of (1) by $\vec{b}\times\vec{c}$

$$\vec{r}\cdot(\vec{b}\times\vec{c}) = l\vec{a}\cdot(\vec{b}\times\vec{c}) + m\vec{b}\cdot(\vec{b}\times\vec{c}) + n\vec{c}\cdot(\vec{b}\times\vec{c})$$

$$= l[\vec{a}\,\vec{b}\,\vec{c}] + m[\vec{b}\,\vec{b}\,\vec{c}] + n[\vec{c}\,\vec{b}\,\vec{c}]$$

$$= l[\vec{a}\,\vec{b}\,\vec{c}] + 0 + 0$$

$$\therefore \vec{r}\cdot(\vec{b}\times\vec{c}) = l[\vec{a}\,\vec{b}\,\vec{c}]$$

~~Similarly,~~

$$\text{or, } l = \frac{\vec{r}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$\therefore l = \vec{r}\cdot\vec{a}'$$

Similarly

$$m = \vec{r}\cdot\vec{b}'$$

$$n = \vec{r}\cdot\vec{c}'$$

substituting the values of l, m, n in eq^n. (1)

$$\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}$$

Proved

27. Find a set of vectors reciprocal system to the set of vectors $2\vec{i}+3\vec{j}+\vec{k}$, $-\vec{i}+2\vec{j}-3\vec{k}$ and $3\vec{i}-4\vec{j}+2\vec{k}$. [2073, 2075]

⇒ Solution,

Let, $\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b} = -\vec{i}+2\vec{j}-3\vec{k}$ and $\vec{c} = 3\vec{i}-4\vec{j}+2\vec{k}$

Now,

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix}$$

$$= 2(6-12)+3(-9+2)+1(6+4)$$

$$= -12 - 21 + 2$$

$$= -35$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 2 & -3 \end{vmatrix}$$

$$= -7\vec{i}+7\vec{j}+7\vec{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix}$$

$$= -8\vec{i}+7\vec{j}-2\vec{k}$$

$$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= -10\vec{i}+\vec{j}+17\vec{k}$$

Hence, the reciprocal system of vectors is

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{1}{35}(8\vec{i}+7\vec{j}+2\vec{k})$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{1}{35}(10\vec{i}-\vec{j}-17\vec{k})$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{1}{35}(7\vec{i}-7\vec{j}-7\vec{k}).$$