Unit-2 Straight line

→ 1 S.Q., 1 L.Q → 11 marks

Q-1 Find the eqⁿ of straight line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2). Find the point where the line joining $(2,1,3)$ and $(4,-2,5)$ cuts the plane $2x+y-z-3=0$. [2062, 2067, 2072, 2069 ...]

⇒ Solution,

The eqⁿ of straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \longrightarrow (1)$$

Where l, m, n are the d.c.s. of the line.

Since, line (1) passes through (x_2, y_2, z_2)

$$\frac{x_2-x_1}{l} = \frac{y_2-y_1}{m} = \frac{z_2-z_1}{n} \longrightarrow (2)$$

Eliminating l, m, n from eqⁿ (1) and (2)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Which is required eqⁿ of the straight line.

Next-part

The eqⁿ of line passing through $(2,1,3)$ and $(4,-2,5)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

i.e. $\frac{x-2}{4-2} = \frac{y-1}{-2-1} = \frac{z-3}{5-3}$

or, $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z-3}{2}$ ——→ (1)

Let,

$\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z-3}{2} = r$ (say)

Then the co-ordinate of any point in the line (1) is

$(2r+2, -3r+1, 2r+3)$

Since the line (1) cuts the given plane

$2x+y-z-3=0$

So, $4r+4-3r+1-2r-3-3=0$

or, $-r-1=0$

$\therefore r=-1$

Hence, the required points are

$(2r+2, -3r+1, 2r+3)$

or $(-2+2, +3+1, -2+3)$

or, $(0, 4, 1)$

Q 2→ Obtain the expression for the angle b/w a line with d.r.s l, m, n and a plane $ax+by+cz+d=0$. Find the points at which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ cuts a surface $11x^2-5y^2+z^2=0$ [2072]

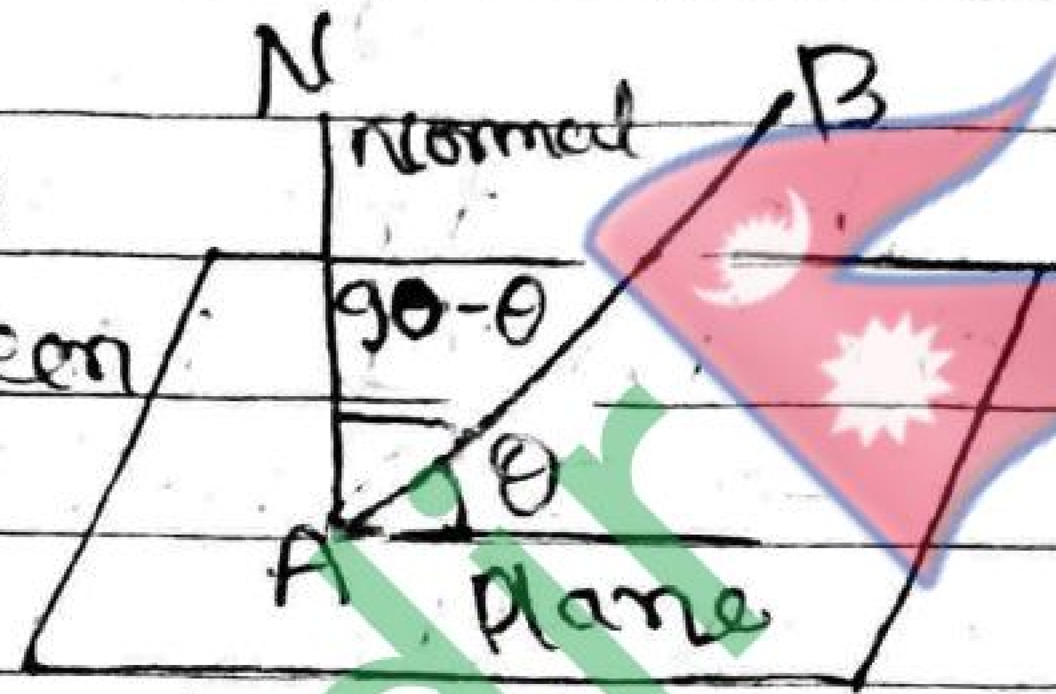
⇒ Solution,

Let the eqn of straight line with d.r.s l, m, n through point (x_1, y_1, z_1) be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

and the plane

$ax + by + cz + d = 0 \longrightarrow (2)$



If θ be the angle between the given line (1) and the given plane (2). Then the angle between the given line (1) and the normal to the plane (2) is $(90-\theta)$

Therefore, the d.r.s. of the normal to the plane (2) are a, b, c.

$\therefore$ The direction cosines of normal are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

The d.c.s. of the given straight line are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

Since the angle between the line (1) and normal to the plane (2) is $(90^\circ-\theta)$

$$\therefore \cos(90^\circ-\theta) = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

or, $$\sin\theta = \frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}} \longrightarrow (3)$$

This gives the required angle between straight (1) and plane (2).

Next-part

Let, $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2} = r$ (say) $\rightarrow (1)$

Then the co-ordinates of any point on the line (1) is

$(-r-1,\ 5r+12,\ 2r+7)$

Since, line (1) cuts the surface $11x^2-5y^2+z^2=0$ at this point then, we have

$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$

or, $+11(r^2+2r+1) - 5(25r^2+120r+144) +$

or, $4r^2+28r+49=0$

or, $11r^2+22r+11-125r^2-600r-720+4r^2+28r+49=0$

or, $-110r^2-550r-660=0$

or, $r^2+5r+6=0$

or, $r^2+3r+2r+6=0$

or, $r(r+3)+2(r+3)=0$

or, $(r+2)(r+3)=0$

$\therefore r=-2,-3$

Hence, the required co-ordinates of any point on the line (1) is $(2,-3,1)$ and $(1,2,3)$.

Q-13 Find the equation of plane containing line $2x-5y+2z-6=0$, $2x+3y-z-5=0$ and parallel to the line $\frac{x}{1}=\frac{y}{-6}=\frac{z}{7}$

$\Rightarrow$ Solution, [B.A/BSc. 20

The eq^n of plane containing the line $2x-5y+2z-6=0$ and $2x+3y-z-5=0$ is

$(2x-5y+2z-6)+\lambda(2x+3y-z-5)=0$

or, $(2+2\lambda)x+(3\lambda-5)y+(2-\lambda)z-(6+5\lambda)=0 \longrightarrow (1)$

If plane (1) is parallel to the given line $\frac{x}{1}=\frac{y}{-6}=\frac{z}{7}$ the d.r.s of the line normal to plane (1) are $2+2\lambda$, $3\lambda-5$, $2-\lambda$ then using perpendicularity condition.

$(2+2\lambda)1+(3\lambda-5)(-6)+(2-\lambda)7=0$

or, $2+2\lambda-18\lambda+30+14-7\lambda=0$

or, $\lambda=+2$

Substituting the value of λ in eq^n (1), we get

$(2+4)x+(6-5)y+0-(6+10)=0$

or, $6x+y-16=0$

Which is the required eq^n of plane

P.T.O

Q→4 Find the equation of plane through point (f, g, h) and parallel to the lines $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$ and $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$. [2064]

⇒ Solution,

The equation of given lines are

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \longrightarrow (1)$$

and

$$\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \longrightarrow (2)$$

The equation of plane through the point (f, g, h) is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

or, $a(x-f) + b(y-g) + c(z-h) = 0 \longrightarrow (3)$

Where a, b, c are the d-c-s. of the normal to plane (3).

If the plane (3) parallel to the line (1) and (2), then their normals are perp

So,

$$al_1 + bm_1 + cn_1 = 0 \longrightarrow (4)$$

$$al_2 + bm_2 + cn_2 = 0 \longrightarrow (5)$$

Solving eqn. (4) and (5) by cross multiplication method; we get

$$\frac{a}{m_1n_2-m_2n_1}=\frac{b}{n_1l_2-n_2l_1}=\frac{c}{l_1m_2-l_2m_1}=K \text{ (say)}$$

Substituting this values of a, b, c in eqn. (3) we get

$$(m_1n_2-m_2n_1)(x-f)+(n_1l_2-n_2l_1)(y-g)+(l_1m_2-l_2m_1)(z-h)=0$$

Which is required equation of the plane.

5 Find the equation of plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ parallel to the line $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$ [2066]

⇒ Solution,

~~Given eqn. of plane is~~

$$\cancel{\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}} \longrightarrow (1)$$

The eqn. of the plane through the given line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0 \longrightarrow (1)$$

where a, b, c are the d.r.s. of the line normal to the plane (1), So, $al + bm + cn = 0 \longrightarrow (2)$

plane (1) is parallel to line (2) is

$$\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}$$

So, $al' + bm' + cn' = 0 \longrightarrow (3)$

Solving (2) and (3) by cross multiplication rule, we get

$$\frac{a}{mn' - m'n} = \frac{b}{nl' - n'l} = \frac{c}{lm' - l'm}$$

Substituting this values of a, b, c in eqn. (1) we get,

$$(mn' - m'n)(x - \alpha) + (nl' - n'l)(y - \beta) + (lm' - l'm)(z - \gamma) = 0$$

Which is required equation of plane.

6. Find the equation of the line through the point (2, 3, 1) and parallel to the planes $2x + 3y + 4z = 5$ and $3x + 4y + 5z = 6$ [2069]

$\Rightarrow$ Solution,

Eqn of the line through the point (2, 3, 1) is

$$\frac{x-2}{l} = \frac{y-3}{m} = \frac{z-1}{n} \longrightarrow (1)$$

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Since line (1) is parallel to the plane
$2x+3y+4z=5 \longrightarrow (2)$
The d.r.s. of the line normal to plane (2) are, 2, 3, 4.
So, Using perpendicularity condition
$2l+3m+4n=0 \longrightarrow (3)$
Also, line (1) parallel to the plane
$3x+4y+5z=6 \longrightarrow (4)$
The d.r.s of the line normal to the plane (4) are 3, 4, 5
Using perpendicularity condition,

$3l+4m+5n=0 \longrightarrow (5)$
Now solving eq^n (3) and (5) by cross multiplication rule, we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = K$ (say)
Substituting these values of l, m, n in eq^n (1), we get

$$\frac{x-2}{-1} = \frac{y-3}{2} = \frac{z-1}{-1}$$

or, $\frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{1}$ ##

Find the equation of the plane containing the line $\frac{x-1}{2} = \frac{-y-1}{2} = \frac{z-3}{4}$ and perp. to the plane $x+2y+2z = 12$. [T.U. 2057]

⇒ Solution,

Given eq^n of the line

$$\frac{x-1}{2} = \frac{-y-1}{2} = \frac{z-3}{4}$$

or, $$\frac{x-1}{2} = \frac{y+1}{-2} = \frac{z-3}{4}$$

The d.r.s. of the line are $2, -2, 4$

Eq^n of the plane containing the given line is

$$a(x-1) + b(y+1) + c(z-3) = 0 \longrightarrow (1)$$

Where,

$2a - 2b + 4c = 0$

$a - b + 2c = 0 \longrightarrow (2)$

If plane (1) is perp. to the plane $x + 2y + 2z = 12$.

Then,

$a + 2b + 2c = 0 \longrightarrow (3)$

Solving eq^n (3) and (2) by cross multiplication rule,

we get

$$\frac{a}{-2-4} = \frac{b}{2-2} = \frac{c}{2+1}$$

or, $\dfrac{a}{-6} = \dfrac{b}{0} = \dfrac{c}{3}$

or, $\dfrac{a}{-2} = \dfrac{b}{0} = \dfrac{c}{1} = k$ (say)

Substituting these values values of a, b, c in eqn. (1), we get

$-2(x-1)+0+1(z-3)=0$

or, $-2x+2+z-3=0$

or, $2x-z+1=0$

which is required equation of plane.

8. Find the equation of plane containing the line $\dfrac{x-1}{2} = \dfrac{-y-1}{1} = \dfrac{z-3}{4}$ and is perp. to the plane $x+2y+$ [2057]

⇒ Solution,

Given, eqn. of the line

$$\frac{x-1}{2} = \frac{-y-1}{1} = \frac{z-3}{4} \longrightarrow (1)$$

The d.r.s of the line are 2, -1, 4
so, eqn. of the plane containing the given line is

$$a(x-1)+b(y+1)+c(z-3)=0 \longrightarrow (2)$$

Where,

$$2a-b+4c=0 \longrightarrow (3)$$

If the plane (2) is perp to the plane $x+2y+z=12$
then

$$a+2b+c=0 \longrightarrow (4)$$

Solving eqn. (3) and (4), we get

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1}$$

or, $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5} = K \text{ (say)}$

Substituting these values of a, b, c in eqn. (2), we get

$$-9(x-1)+2(y+1)+5(z-3)=0$$

or, $9x+2y-5z+4=0$

Which is required eqn of plane.

9. Under what condition, a given line is parallel to a given plane. Also write the condition that the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ may lie in the plane $ax+by+cz+d=0$. Find the equation of the line through (α, β, γ) and parallel to the plane $lx+my+nz=p$ and $l'x+m'y+n'z=p'$.
[2054 TU]

⇒ Solution,
Let the given line and the plane be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \longrightarrow (1)$$

and $ax+by+cz+d=0 \longrightarrow (2)$

Since given line (1) is parallel to the given ~~line~~ plane (2), the normals of plane (2) is perp. to line (1).
So, using perpendicularity condition,

$a \cdot l + b \cdot m + c \cdot n = 0$

or, $al + bm + cn = 0$

Which is required condition.

Next part

The condition that the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ may lie in the plane $ax+by+cz+d=0$ is

$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$
where, $al+bm+cn=0$

Last part
The given equation of planes are
$lx+my+nz=p \longrightarrow (1)$
& $l'x+m'y+n'z=p' \longrightarrow (2)$

The equation of the line through point (α, β, γ) is
$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c} \longrightarrow (3)$$
Where a, b, c are the d.r.s of the line (3).
If line (3) is parallel to line (1) and (2), so using perpendicularity condition.

$la+mb+nc=0 \longrightarrow (4)$
$l'a+m'b+n'c=0 \longrightarrow (5)$
Solving eqn. (4) and (5), we get
$$\frac{a}{mn'-m'n}=\frac{b}{nl'-n'l}=\frac{c}{lm'-l'm}=K \text{ (say)}$$

Which gives us

$a = (mn' - m'n)K$
$b = (nl' - n'l)K$
$c = (lm' - l'm)K$

Now, eqn. (3) becomes

$$\frac{x-\alpha}{mn'-m'n} = \frac{y-\beta}{nl'-n'l} = \frac{z-\gamma}{lm'-l'm}$$

which is the required equation.

10. Find the condition that the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ may lie the plane $ax+by+cz+d=0$. Show that the line $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ lies in the plane $3x+4y-5z-13=$

⇒ Solution

Let given eqn. of line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \longrightarrow (1)$$

and plane is

$$ax+by+cz+d=0 \longrightarrow (2)$$

Let $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ (say)

Then its coordinates are
$(lr+x_1, mr+y_1, nr+z_1)$,

If the line (1) lies on the plane (2), then the plane must contain that point. So this coordinates lies on the plane (2)

$\therefore\ a(lr+x_1)+b(mr+y_1)+c(nr+z_1)+d = 0$

or, $r(al+bm+cn)+(ax_1+by_1+cz_1+d)=0 \quad (3)$

The relation (3) is true for all values of r.
Therefore,
$al+bm+cn=0$ and
$ax_1+by_1+cz_1+d=0$
which are required equations.

Next part
Given line is
$$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5} \longrightarrow (1)$$
and plane is
$$3x+4y-5z+3=0 \longrightarrow (2)$$
Line (1) passes through (2, 4, 5) and its d.r.s is 3, 4, 5. Since, line

(1) lies on plane (2),
so, normal of plane (2) is perp to line (1).

$al + bm + cn = 0$

or, $3.3 + 4.4 - 5.5 = 0$

or, $0 = 0$ (True)

Also, $ax_1 + by_1 + cz_1 + d = 0$

or, $3\cdot2 + 4\cdot4 - 5\cdot5 + 3 = 0$

or, $0 = 0$ (True)

Hence, the line (1) lies on plane (2)

11. Find the equation of plane containing the line $\frac{x-1}{2} = \frac{-y-1}{1} = \frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z = 12$ [TU 2057]

⇒ solution,

Given equation of the line is

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \longrightarrow (1)$$

and plane is

$$x+2y+z = 12 \longrightarrow (2)$$

The equation of the plane containing the line (1)

$$a(x-1) + b(y+1) + c(z-3) = 0 \longrightarrow (3)$$

Where a, b, c are the d.r.s of the line normal to plane (3). Also, d.r.s of the line (1) are 2, −1, 4. So, using perpendicularity condition

$2a - b + 4c = 0 \longrightarrow (4)$

If the plane (3) is perp. to plane (2) ~~is~~ then, using perpendicularity condition

$a + 2b + c = 0 \longrightarrow (5)$

Solving eqn. (4) and (5), we get

$\frac{a}{-9} = \frac{b}{2} = \frac{c}{5} = k$ (say)

Which gives us,

$a = -9k,\ b = 2k,\ c = 5k$

Now eqn. (3) becomes,

$-9(x-1) + 2(y-1) + 5(z-3) = 0.$

or, $9x - 2y - 5z + 4 = 0.$

Which is the required equation.

Coplanar Lines

12. What are coplanar lines? write the condition for two straight line in symmetrical form to be coplanar. prove that the lines
$$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta} \text{ and}$$
$$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b+c}{\beta+\gamma} \text{ are}$$
coplanar and find the plane in which they lie. [2054, 2061]

⇒ Solution,

Coplanar lines

System of lines are said to be coplanar if a plane can be drawn parallel to all of them.

Condition of two straight lines is Symmetrical form to be coplanar.

Let the given two straight lines in symmetrical form be

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and}$$
$$\frac{x-x_2}{l_2} = \frac{y-y_2}{n_2} = \frac{z-z_2}{n_2}$$

The required condition for the given lines to be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Next part

The given lines are

$$\frac{x-(a-d)}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{\alpha+\delta}$$

and

$$\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}$$

If these two lines will be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Applying $C_1 \longrightarrow C_1 + C_3$

$$\begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

or, $$2\begin{vmatrix} b-a & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

or, $2 \times 0 = 0$

or, $0 = 0$

Hence, the given two lines are coplanar.

Again,

The required eqn. of plane in which both given lines lie is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\text{or, } \begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta-\gamma \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_2$ & $C_3 \to C_3 - C_2$

$$\text{or, } \begin{vmatrix} x+d-y & y-a & z-y-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0$$

or, $(x+d-y)(\alpha\gamma-\beta\delta) - (y-a)(\cancel{-\delta\gamma}+\cancel{\delta\gamma})$
$+(z-y-d)(-\delta\beta+\alpha\gamma) = 0$

~~or, $\alpha\gamma x + \alpha\gamma d - \alpha\gamma y - \beta\delta x - \beta\delta d + \beta\delta y$ $= 0$ $- \delta\beta z + \delta\beta y + \delta\beta d + \alpha\gamma z - \alpha\gamma y - \alpha\gamma d = 0$~~

or. $(\alpha\gamma-\beta\delta)(x+\cancel{d}-y+z-y-\cancel{d}) = 0$

or, $(\alpha\gamma-\beta\delta)(x+z-2y) = 0$

Since $\alpha\gamma - \beta\delta \neq 0$

$$x-2y+z=0$$

which is required plane.

13. Prove that the lines $\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'}$ and $\frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$ intersect and find the coordinates of the point of intersection. [2054]

⇒ Solution,

Given two lines are

$$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'} \longrightarrow (1)$$

and

$$\frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c} \longrightarrow (2)$$

Now,

$$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'} = r \text{ (say)}$$

So any point on the line (1) is $(a'r+a, b'r+b, c'r+c)$

Also,

$$\frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c} = r' \text{ (say)}$$

So any point on the line (2) is $(ar'+a', br'+b', cr'+c')$

Since the given two lines intersect at a point, so these two points are equal each other.

$a'r+a = ar'+a'$ or, $ar' - a'r = a - a'$ (3)

$b'r + b = br' + b'$ or, $br' - b'r = b - b' \rightarrow (4)$
$c'r + c = cr' + c'$ or, $cr' - c'r = c - c' \rightarrow (5)$

Solving equation (3) and (4),

$a\cancel{b}r' - a'br = a\cancel{b} - a'b$
$\cancel{ab}r' - ab'r = ab - ab'$
$- \quad + \quad - \quad +$

$r(ab' - a'b) = ab' - a'b$

$\therefore r = 1$

Putting this value of r in eqn. (3)

$ar' - \cancel{a'} = a - \cancel{a'}$
or, $r'a = a$

$\therefore r' = 1$

Since the values of r and r' also satisfied given lines intersect to each other.
Also
Point of intersection of the given two lines (1) and (2) is
$(a' + a, b' + b, c' + c)$

14. State the condition for two straight lines in general form to be coplanar. Prove that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and

$$\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$ are coplanar. Also find the eqⁿ of plane in which they lie.

⟹ Solution,
Let the given eqⁿ of straight lines in general form is

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$
and
$$\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$$

These two lines will be coplanar if

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$$

Next part
The given two lines are

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} \quad \text{and}$$

$$\frac{x-0}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$

If these two lines will be coplanar then,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\text{or, } \begin{vmatrix} 0+1 & 7-3 & -7+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

$$\text{or, } \begin{vmatrix} 1 & 4 & -5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or, $1(4+3) + 4(+1+6) - 5(9-2) = 0$

or, $7 + 28 - 35 = 0$

or, $0 = 0$ (True)

Hence, proved

Last part

Let eqn of plane containing the given line is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or, $(x+1)(4+3)+(y-3)(1+6)+(z+2)(9-2)$

or, $7(x+1)+7(y-3)+7(z+2)$

or, $7(x+y+z)=0$

$\therefore x+y+z=0$

which is the required eqn of plane.

15. Prove that the lines $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}$ and $3x+2y+z-2=0=x-3y+2z-13$ are coplanar [2056, 2059, 2061/68]

⇒ solution,

The eqn. of given lines are

$$\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2} \longrightarrow (1)$$

and

$3x+2y+z-2=0=x-3y+2z-13 \longrightarrow (2)$

Equation of plane through the line (2) is

$3x+2y+z-2+\lambda(x-3y+2z-13)=0$

where λ is constant

or, $(3+\lambda)x+(2-3\lambda)y+(1+2\lambda)z-(2+13\lambda)=0 \longrightarrow (3)$

Direction cosines of the line normal to plane (3) are proportional to $(3+\lambda, 2-3\lambda, 1+2\lambda)$

Since, plane (3) is parallel to line (1), using perpendicularity condition

$(3+\lambda)3+(2-3\lambda)1+(1+2\lambda)(-2)=0$

or, $9+3\lambda+2-3\lambda+(-2-4\lambda)=0$

or, $\lambda=9/4$

Now, eqn. (3) becomes,

$3x+2y+z-2+\frac{9}{4}(x-3y+2z-13)=0$

or, $12x+8y+4z-8+9x-27y+18z-117=0$

or, $21x-19y+22z-125=0 \longrightarrow (4)$

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Any point on the line (1) is $(-5,-4,7)$
Since, $(-5,-4,7)$ lies on the plane (4)

$21\times(-5) - 19\times(-4) + 22\times 7 - 125 = 0$
or, $0 = 0$ (True)

Hence the plane (4) contains line (1) and (2).
So, given two lines are coplanar and the equation of plane in which they lie is
$21x - 19y + 22z - 125 = 0$.

16. What is coplanar lines? prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplanar. Also find the point of intersection [2063, 2071]

$\Rightarrow$ Solution,
Let the given lines are
$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$ (say) $\longrightarrow$ (1)
and
$4x-3y+1=0=5x-3z+2 \longrightarrow$ (2)

The coordinate of any point on line (1) is

$(2r+1, 3r+2, 4r+3)$
The lines (1) and (2) will be coplanar if they intersect at a point. Since line (2) will be obtain by the intersection of two planes

$4x-3y+1=0 \longrightarrow (3)$
and $5x-3z+2=0 \longrightarrow (4)$

If the equation of given lines (1) and intersect at a point since there must be a common point between the line (1) and plane (3). So the point $(2r+1, 3r+2, 4r+3)$ lies on plane (3)

$4(2r+1)-3(3r+2)+1=0$
or, $8r+4-9r-6+1=0$
or, $r=-1$
Hence the line (1) meets the plane (3) at $(-2+1, -3+2, -4+3) = (-1,-1,-1)$
The point $(-1,-1,-1)$ clearly satisfies the another plane (4)
$5(-1)-3(-1)+2=0$
or, $-5+5=0$
or, $0=0$ (True)
Thus two lines (1) and (2) intersect at point $(-1,-1,-1)$. Hence they are coplanar.

17. Show that the lines $x+y+z-3=0=2x+3y+4z-5$ and $4x-y+5z-7=0=2x-5y-z-3$ are coplanar. Find the plane in which they lie. [2073]

⇒ Solution,

The given line is

$x+y+z-3=0=2x+3y+4z-5 \longrightarrow (1)$

Let l, m, n be the d.r.s. of the line (1). Then

$l+m+n=0$ and $2m+3n+4n=0$

Solving these by cross multiplication rule,

$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$

∴ The d.r.s of the line (1) are $1, -2, 1$.

Let the line (1) meets the plan $x=0$ at $(0, y_1, z_1)$

Then,

$y_1+z_1=3$ and $3y_1+4z_1=5$

Solving we get

$y_1=7, z_1=-4$

So the line (1) meets the plan $x=0$ at $(0, 7, -4)$

Hence, in the symmetrical form

$\frac{x}{1} = \frac{y-7}{-2} = \frac{z+4}{1}$

Also, the given line is

$4x - y + 5z - 7 = 0 = 2x - 5y - z - 3 \longrightarrow (3)$

Now, general equation of the plane through (3) is

$(4x - y + 5z - 7) + \lambda(2x - 5y - z - 3) = 0 \quad (4)$

where λ is a constant.

or, $(4 + 2\lambda)x - (7 + 5\lambda)y + (5 - \lambda)z - (3\lambda + 7) = 0$

This will be parallel to line (2) if

$1(4 + 2\lambda) + 2(7 + 5\lambda) + 7(5 - \lambda) = 0$

or, $4 + 2\lambda + 2 + 10\lambda + 5 - \lambda = 0$

or, $11\lambda = 11$

$\lambda = -1$

Now, eqn (4) becomes,

$4x - y + 5z - 7 - 1(2x - 5y - z - 3) = 0$

or, $4x - y + 5z - 7 - 2x + 5y + z + 3 = 0$

or, $x + 2y + 3z - 2 = 0$

which is required plane.

The plane passes through $(0, 7, -4)$

$0 + 14 - 12 - 2 = 0$

or, $0 = 0$ (True)

Hence given two lines are coplanar.

Shortest distance

18. Define shortest distance betn. the lines. Obtain the eqn. of the line of shortest distance between the lines. (2069)

⇒ Solution,
The perpendicular distance between the no-intersecting lines and non-parallel lines (skew lines) is known as the line of shortest distance and it is denoted by S.D.

Next part :
Let the given equation of two skew lines AB and CD is

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \longrightarrow (1)$$

and

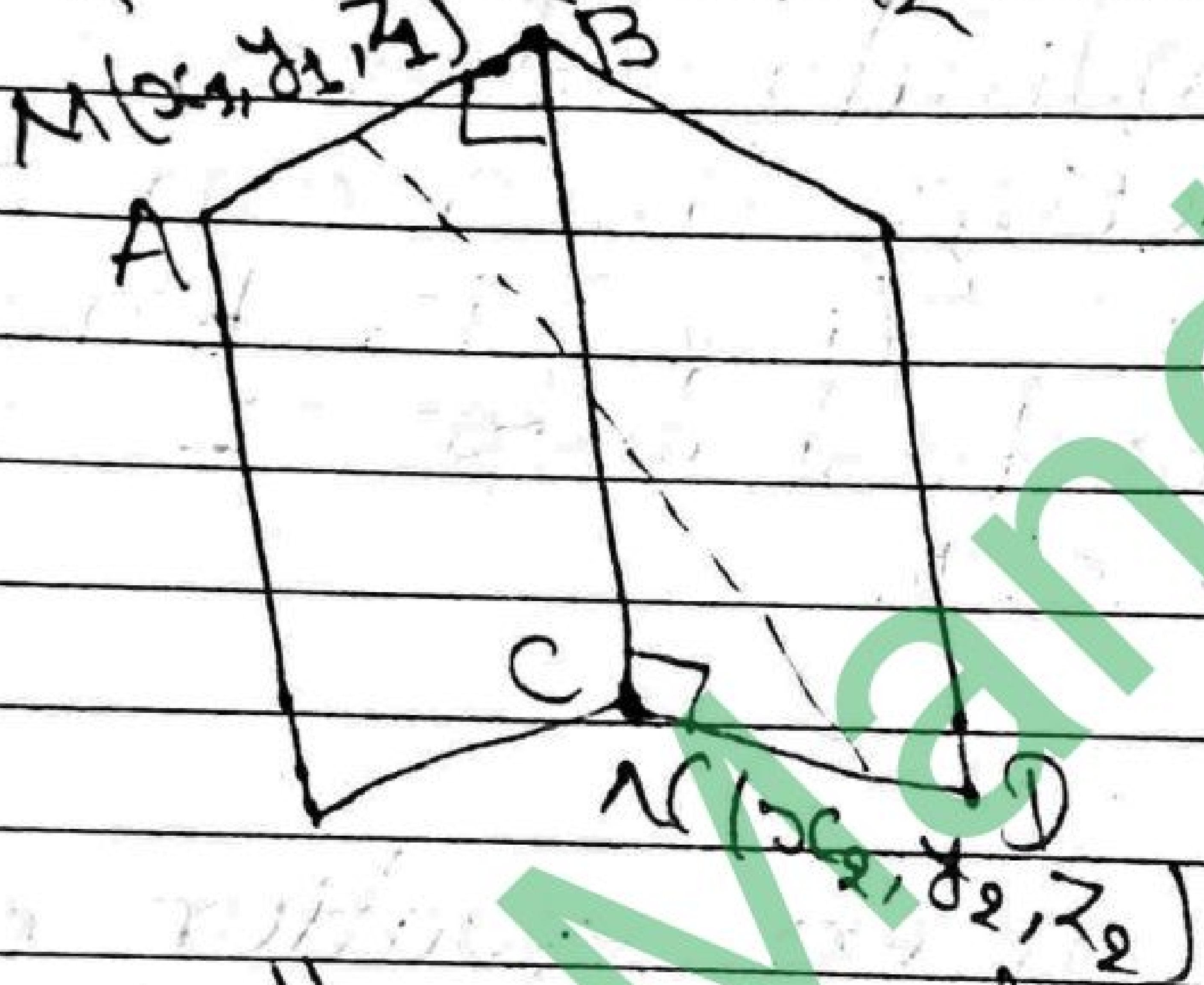
$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \longrightarrow (2)$$

The lines AB and CD pass through the points $M(x_1, y_1, z_1)$ and $N(x_2, y_2, z_2)$ respectively.
Let BC be perpendicular to both lines AB and CD. Then BC be the shortest distance and it's d.r.s is l, m, n. Since

BC is perpendicular to the lines (1) and (2). Using perpendicularity condition we get

$ll_1 + mm_1 + nn_1 = 0$

and $ll_2 + mm_2 + nn_2 = 0$



Solving these equations by cross multiplication rule,

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Therefore, d.c.s of the shortest distance are,

$$\frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

Now, BC is the length of shortest distance is the projection of MN on BC

∴ Length of shortest distance is BC

= projection of MN on BC

$$= (x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$$

where l, m, n are ~~the~~ the actual d.c s of BC.

The length of shortest distance

$$= \frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

$$= \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}$$

The line BC may be obtained by the intersection of AB, BC and CD, BC.

The equation of the containing line CD and the equation of the shortest distance BC is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \longrightarrow (\mathrm{II})$$

Also, the equation of the plane containing the line CD and the equation of the line of shortest distance BC is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_1 \\ l_e & m_e & n_e \\ l & m & n \end{vmatrix} = 0 \longrightarrow (4)$$

The eqn of planes (3) and (4) taken ~~over~~ together is the required eqn of S.D.

19 Find the shortest distance between the lines $\frac{x+1}{2} = \frac{y-3}{4} = \frac{z-2}{5}$ & $\frac{x-2}{3} = \frac{y+3}{-2} = \frac{z-4}{1}$. Also, find the equation of the line of shortest distance. [2070]

⇒ solution,

The given equation of the straight lines are

$$\frac{x+1}{2} = \frac{y-3}{4} = \frac{z-2}{5} \longrightarrow (1)$$

& $$\frac{x-2}{3} = \frac{y+3}{-2} = \frac{z-4}{1} \longrightarrow (2)$$

The eqn of plane containg the line

(1) and parallel to line (2) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x+1 & y-3 & z-2 \\ 2 & 4 & 5 \\ 3 & -2 & 1 \end{vmatrix} = 0$$

or, $(x+1)(4+10)+(y-3)(15-2)+(z-2)(-4-12) = 0$

or, $14x+14+13y-39-16z+32=0$

or, $14x+13y-16z+7=0 \longrightarrow (3)$

Any point on the line (2) is 2, -3, 4

Length of shortest distance = perp. distance from (2, -3, 4) to plane (3)

$$= \left| \frac{14\times2+13(-3)-16\times4+7}{\sqrt{14^2+13^2+(-16)^2}} \right|$$

$= \dfrac{68}{3\sqrt{69}}$ unit #

Then, eqn of plane containing the line (1) and perp. to the plane (3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

direction cosines of the line normal to plane (3) are 14, 13, -16. So

$$\begin{vmatrix} x+1 & y-3 & z-2 \\ 2 & 4 & 5 \\ 14 & 13 & -16 \end{vmatrix} = 0$$

or, $(x+1)(-129) + (y-3)(102) + (z-2)(-30) = 0$

or, $129x - 102y + 30z + 375 = 0 \longrightarrow (4)$

Also,
the eqn of the plane containing line (2) and perp. to the plane (3) is

$$\begin{vmatrix} x-2 & y+3 & z-4 \\ 3 & -2 & 1 \\ 14 & 13 & -16 \end{vmatrix} = 0$$

or, $(x-2)(32-13) + (y+3)(14+48) + (z-4)(39+28) = 0$

or, $19x + 62y + 67z - 120 = 0 \longrightarrow (5)$

Thus, from eqn. (4) and (5) the eqn. of shortest distance is

$129x - 102y + 30z + 375 = 0 = 19x + 62y + 67z - 120.$

20. Show that the shortest distances between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ &

$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is $\frac{1}{\sqrt{6}}$. Also find the equation of S.D. [2060, 2064]

$\Rightarrow$ Solution,

Given straight lines are

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\longrightarrow (1)$

$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \longrightarrow (2)$

The equation of plane containing line (1) and parallel to (2) is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$(x-1)(15-16) + (y-2)(12-10) + (z-3)(8-9)$

or, $x - 2y + z = 0 \longrightarrow (3)$

Any point lie on eqn (2) is 2, 4, 5

∴ S.D = perp. distances from (2, 4, 5) to plane (3)

$$= \left| \frac{1\times 2 - 2\times 4 + 1\times 5}{\sqrt{1+4+1}} \right|$$

$$= \frac{1}{\sqrt{6}} \text{ unit}$$

The eqn of plane containing line (1) and perp. to plane (3) is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

or, $(x-1)(3+8) + (y-2)(8-2) + (z-3)(-4-3)$

or, $11x + 2y - 7z + 6 = 0 \longrightarrow (4)$

Also,
The equation of plane containing line (2) and perp. to plane (3)

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

or, $(x-2)(4+10) + (y-4)(5-3) + (z-5)(-6-4)$

or, $7x+y-5z+7=0 \longrightarrow (5)$

Hence, From eqn (4) and (5), eqn of line of S.D is
$11x+2y-7z+6=0=7x+y-5z+7$

21. Find the shortest distance between the lines:- $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z+3}{4}$ and
$\frac{x+3}{4}=\frac{y-2}{5}=\frac{z-1}{3}$ — 2071

⇒ solution,
The given eqn of the lines are

$$\frac{x-2}{2}=\frac{y-1}{3}=\frac{z+3}{4} \longrightarrow (1)$$

& $$\frac{x+3}{4}=\frac{y-2}{5}=\frac{z-1}{3} \longrightarrow (2)$$

The equation of the plane conteining the line (1) and parallel to line (2)

$$\begin{vmatrix} x-2 & y-1 & z+3 \\ 2 & 3 & 4 \\ 4 & 5 & 3 \end{vmatrix}$$

$(x-2)(9-20)+(y-1)(16-6)+(z+3)(10-12)$

or, $-11x + 22 + 10y - 10 - 2z - 6 = 0$

or, $11x - 10y + 2z - 6 = 0 \longrightarrow (3)$

Any point lie on eqn (2) $(-3, 2, 1)$

$\therefore$ S.D = perp. distance from $(-3, 2, 1)$ to plane (3)

$$= \left| \frac{11 \times (-3) - 10 \times 2 + 2 \times 1 - 6}{\sqrt{121 + 100 + 4}} \right|$$

$$= \left| \frac{-57}{\sqrt{225}} \right|$$

$$= \frac{57}{15} \text{ unit}$$

22. Obtain the length and equation of shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $2x - 3y + 27 = 0$, $2y - z + 20 = 0$. [2054]

$\Rightarrow$ Solution,

The given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \longrightarrow (1)$$

& $2x - 3y + 27 = 0 = 2y - z + 20 \longrightarrow (2)$

The equation of the plane through the line (2) is

$2x - 3y + 27 + \lambda(2y - z + 20) = 0$

or, $2x + (2\lambda - 3)y - \lambda z + (20\lambda + 27) = 0 \longrightarrow (3)$

The direction cosines of the line normal to plane (3) are proportional to $2, 2\lambda - 3, -\lambda$.

so, using perpendicularity condition

$3 \cdot 2 - 1 \cdot (2\lambda - 3) + 1(-\lambda) = 0$

or, $6 - 2\lambda + 3 - \lambda = 0$

or, $9 = 3\lambda$

or, $\lambda = 3$

Now substituting the value of λ in eqn (3), we get

$2x + 3y - 3z + 87 = 0 \longrightarrow (4)$

Also, any point lie on the line (1) is $3, 8, 3$

Now,

$$S.D = \left| \frac{2 \cdot 3 + 3 \cdot 8 - 3 \cdot 3 + 87}{\sqrt{4 + 9 + 9}} \right| = \frac{108}{\sqrt{22}} \text{ unit}$$

Next part
The equation of the plane containing the line (1) and perp. to the plane (4) is

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 3 & -3 \end{vmatrix} = 0$$

or, $(x-3)(3-3) + (y-8)(2+9) + (z-3)(9+2) = 0$

or, $11y - 88 + 11z - 33 = 0$

or, $y + z - 11 = 0 \longrightarrow (5)$

Also
The equation of the plane containing line (2) is
$(2x - 3y + 27) + \lambda(2y - z + 20) = 0$

or, $2x + (2\lambda - 3)y - \lambda z + (20\lambda + 27) = 0 \longrightarrow (6)$

If the plane (4) is perp. to plane (6)
Then,
$2 \cdot 2 + (2\lambda - 3) \cdot 3 - \lambda \cdot (-3) = 0$

or, $\lambda = 5/9$

Hence, the equation of the plane containing line (2) and perp. to (4) is

$$\left(2x-3y+27\right)+\frac{5}{9}(2y-z+20)=0$$

or, $18x-17y-5z+343=0 \longrightarrow (7)$

Thus the equation of shortest distance is

$y+z-11=0=18x-17y-5z+343$

23. Find the length and equation of S.D. between the lines $3x-9y+5z=0=x-y-z$ and $6x+8y+3z-13=x+2y+z-7$ [2073, 2075]

$\Rightarrow$ Solution,

Given eq^n of the line is

$3x-9y+5z=0=x+2y-z \longrightarrow (1)$

Let l, m, n be the d.r.s. of the given line.

so, $3l-9m+5n=0$ and $l+m-n=0$

solving these, we get

$$\frac{l}{1}=\frac{m}{2}=\frac{n}{3}$$

The d.r.s of the line (1) are proportional to 1, 2, 3

Since, line (1) passes through origin.

$\therefore$ The eq^n of the line in symmetrical form is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \longrightarrow (2)$

Also the another line is
$6x+8y+3z-13=0=x+2y+z-3 \rightarrow (3)$
The equation of the plane through the line (3) is
$6x+8y+3z-13+\lambda(x+2y+z-3)=0 \rightarrow (4)$

or, $(6+\lambda)x+(8+2\lambda)y+(3+\lambda)z-(13+3\lambda)=0$

This plane is parallel to line (2).
So
$1\cdot(6+\lambda)+2\cdot(8+2\lambda)+3(3+\lambda)=0$

$\therefore \lambda = -\frac{31}{8}$

Then eqn (4) becomes,

$6x+8y+3z-13-\frac{31}{8}(x+2y+z-3)=0$

or, $48x+64y+24z-104-31x-62y-31z+93=0$

or, $17x+2y-7z-11=0 \longrightarrow (5)$

Also, the point (0,0,0) lie on the line (2)

Now,
$S.D. = \left|\frac{0\cdot17+0\cdot2-0\cdot7-11}{\sqrt{289+4+49}}\right|$

$= \frac{11}{\sqrt{342}}$ unit

Again,
Equation of plane containing the line (2) and perp. to (5) is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 17 & 2 & -7 \end{vmatrix} = 0$$

or, $10x - 29y + 16z = 0 \longrightarrow (6)$

Also, eqn. of the plane containing the line (3) and perp. to (5) is

$17(6+\lambda) + 2(8+2\lambda) + (-7).(3+\lambda) = 0$

$\therefore \lambda = -\frac{97}{14}$

Now, from eqn. (4),
$6x+8y+3z-13-\frac{97}{14}(x+2y+z-3)=0$

$\therefore 13x+82y+55z-109 = 0 \longrightarrow (7)$

Hence, required eqn. of S.D. is
$10x-29y+16z = 0 = 13x+82y+55z-109.$